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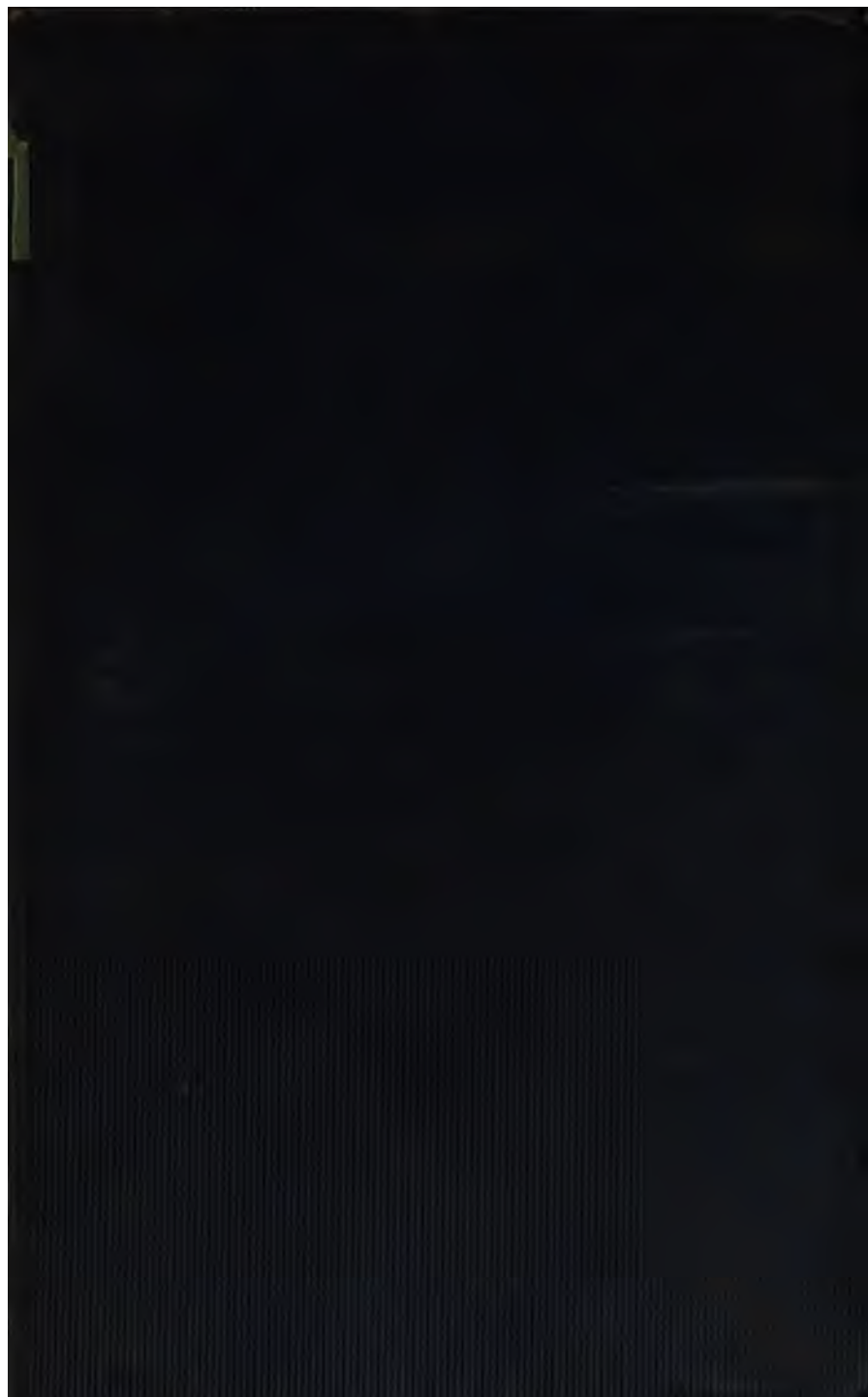
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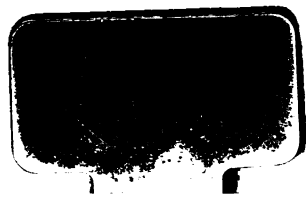
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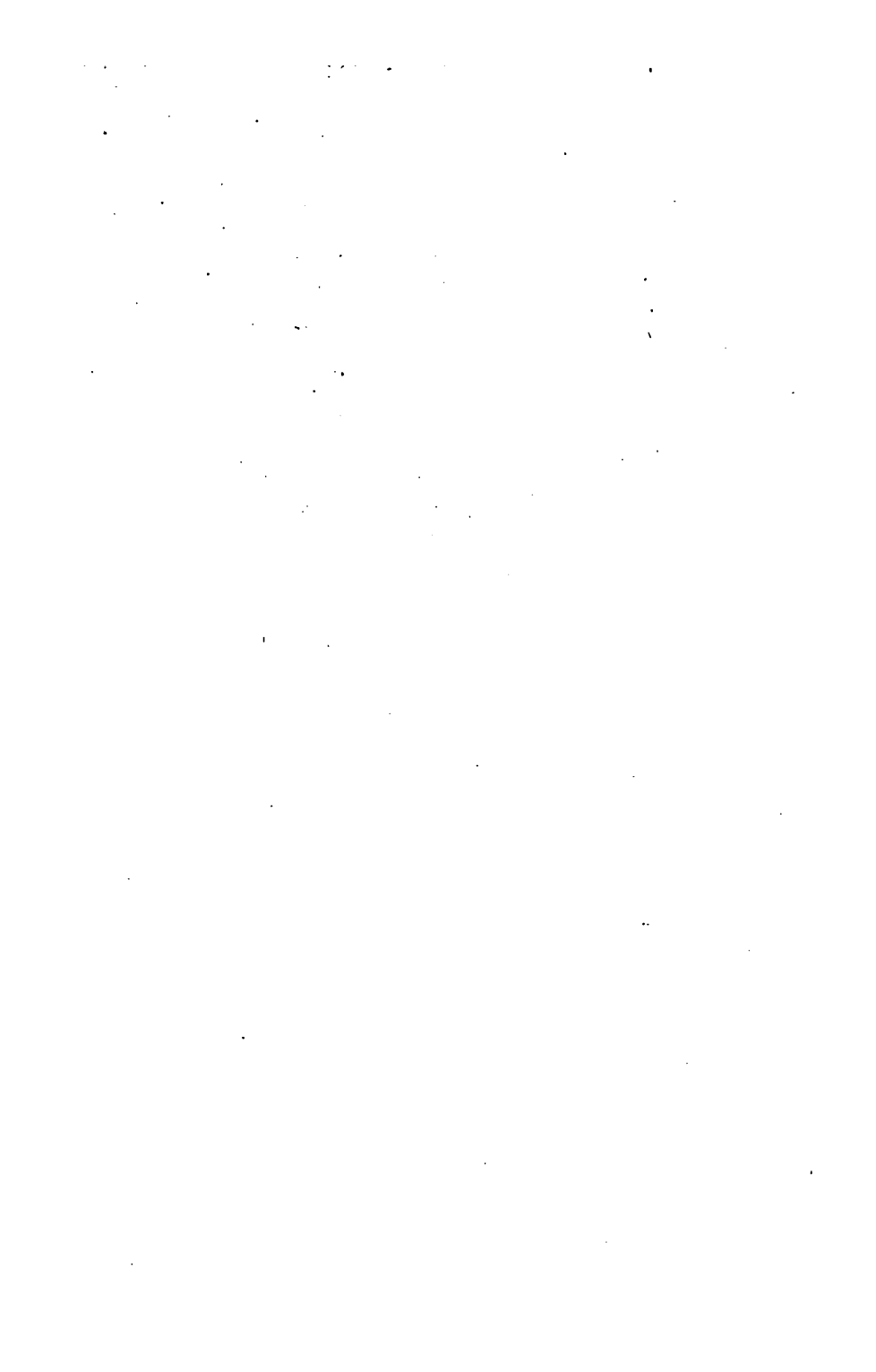
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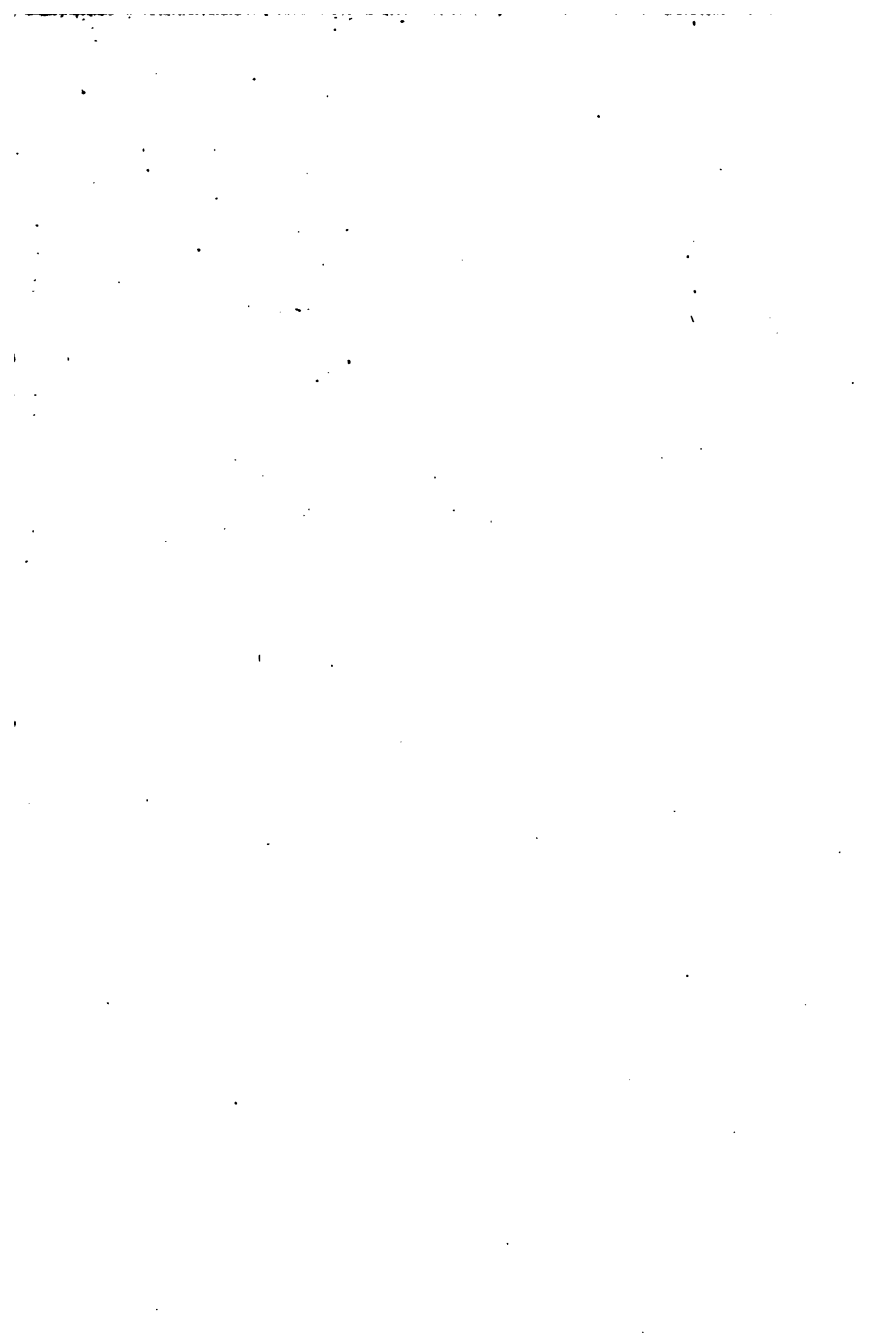


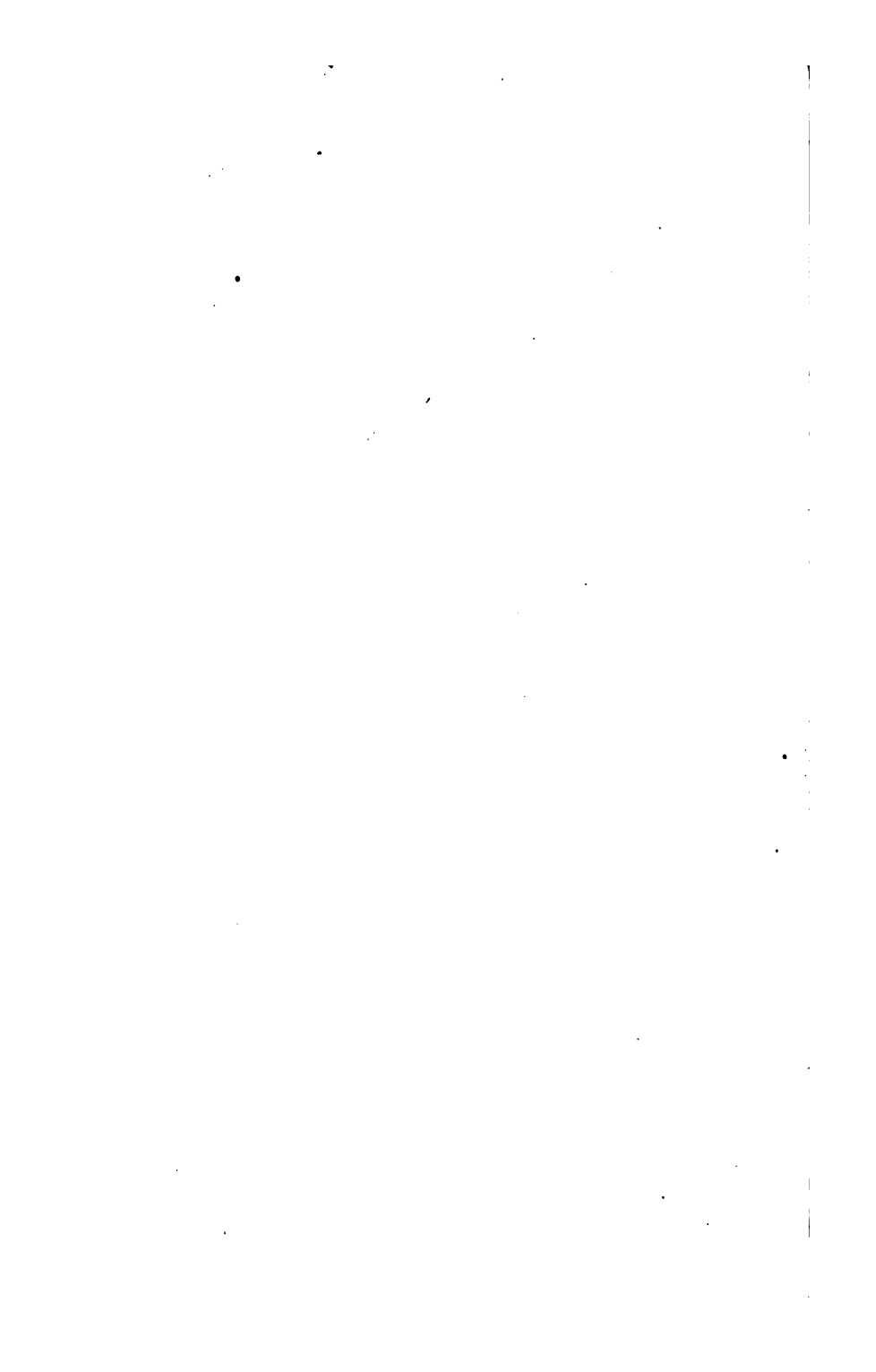


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GEOMETRICAL PROPOSITIONS

DEMONSTRATED;

OR, A

SUPPLEMENT TO EUCLID.

GEOMETRICAL PROPOSITIONS
DEMONSTRATED;

OR, A

SUPPLEMENT TO EUCLID.

BEING

A KEY
TO THE EXERCISES
APPENDED TO EUCLID'S ELEMENTS

BY

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PREFACE.

THE Propositions appended as Exercises to my edition of EUCLID'S ELEMENTS, will be here found demonstrated in such a way that they may be quickly mastered by any one possessing moderate capacity and application. No pains have been spared in their illustration; and the student will perceive, that the more complicated Diagrams here set before him are executed with an unusual degree of neatness. The work itself has been much developed in its plan during its progress towards completion—partly owing to the very encouraging manner in which the edition of the 'Elements' has been received by the public; but still more, to the great fertility of those Geometrical Theorems which it was intended to impress on the mind of the student. In the form of COROLLARIES or of SCHOLIA will be here found several important Propositions not contained in the Appendix to the Elements. A NOTE also has been added on the tentative or experimental methods of solving some interesting Problems, which cannot be reached by strictly geometrical constructions.

If this little volume professed to be nothing more than a HELP TO TEACHERS, it would still be laying claim

to an office, on the utility of which it were needless to expatiate. The mathematical instructor, desirous of exercising his pupils' ingenuity, and of accustoming them to wield their acquired knowledge with ease, must collect for that purpose problems not included in elementary treatises. But the labour of finding the solution or demonstration of these, however light in itself, may yet be incompatible with his occupations. Much time may be lost, even by those well acquainted with *Elemental Geometry*, in the attempt to discover the solution of a new problem. The most acute will often have to beat about for hours, before they hit upon the exact line of argument which leads directly to the proof. Yet so great is the allurements of discovery, and so provoking is it to be baffled in such a case, that the geometrician cannot bring himself to desist from his puzzling constructions, till compelled by head-ache and the midnight clock; even then, the unconquered problem will haunt him in his dreams. To such a one, we beg to observe, that many a valuable hour may be saved, and much weariness avoided, by having recourse to this Key.

The student who believes (and many a one grows up in the opinion) that *Geometry* is comprised in the series of propositions which he has learned in the *Elements*, or who is not aware that the properties of figures are infinite, remains extremely ignorant of the nature and extent of that science. It ought to be adopted as a constant principle in education or in self-cultivation, that a sound mind should always know

the limits of its knowledge: it should be accurately acquainted with the boundaries of its own domain, and know how wide an horizon extends beyond them. Accordingly, it is of no little importance, that one studying the Elements of Geometry should be familiar with the fact, that he does not fully explore a field of science, but only learns a pathway leading to a particular eminence in it. The Supplementary Propositions here presented to him will serve to convince him how numerous are the interesting points of view lying on both sides of that narrow pathway. He thus gets a more adequate idea of the value of his acquirement: he perceives, that though trivial in itself, it leads to a commanding position, while, at the same time, his eyes are opened to the surrounding prospect.

Those who desire to learn Trigonometry, for the sake of its application in Astronomy, Geodæsy, &c., or in general, all who would ascend to the applied Mathematics, will find their labour greatly abridged by a timely acquaintance with this little volume. Many properties of figures, which here, following in natural order, are easily demonstrated, will appear extremely difficult and abstruse when met with incidentally and isolated. A good knowledge of them will prove, to the practical mathematician, a great accession of power. Even in the common business of measuring a field, the surveyor who is thoroughly versed in the properties of figures has great advantages in respect of accuracy and brevity of method. He is better able to

adapt his mode of proceeding in each case to the circumstances of the ground; he finds more means of verification, and derives so much aid from the abundance of his own resources, as to be often enabled to dispense with multiplied observations.

These Propositions might have been easily increased in number; but they seem, at present, quite sufficient for the purpose for which they are intended; and the student who makes himself master of them, will find his sagacity and power of analysis augmented to such a degree, as to render him, in a short time, independent of foreign aids.

W. D. C.

Note.—In the following pages, the references to the theorems contained in this volume, or to the SUPPLEMENTARY PROPOSITIONS, are distinguished by the letter S. prefixed, in this manner: (S. Prop.); those not so distinguished, are made to COOLEY'S edition of EUCLID'S ELEMENTS, the Appendices to which are occasionally cited; and, as a full explanation of the abbreviated notes and symbols employed is given in that work, it was thought needless to repeat it here.

GEOMETRICAL PROPOSITIONS

DEDUCED FROM

THE FIRST SIX BOOKS

OF

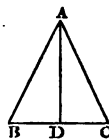
EUCLID'S ELEMENTS.

FIRST BOOK.

PROP. I.

The straight line (AD) which bisects the vertical angle of an isosceles triangle (BAC) bisects the base, and is perpendicular to it.

For since in the triangles BAD, CAD, the sides BA and CA are equal (Hyp.), AD is common to both, and $\angle$ s BAD, CAD, contained by the equal sides, are also equal (Hyp.), the remaining sides or bases BD, DC of the triangles, shall be equal (1. Prop. 4); consequently BC is bisected: and also, since the $\angle$ s ADB, ADC are equal (1. Prop. 4), AD is perpendicular to BC (Def. 7).



PROP. II.

The straight line (AD) which bisects the base of an isosceles triangle (BAC), is perpendicular to it, and bisects the vertical angle.

[See preceding Diagram.]

For since the triangles BAD, CAD, have $AB = AC$ (Hyp.), $BD = DC$ (Hyp.), and AD common, they are equal, and have equal angles opposite to equal sides (I. Prop. 8); consequently, since $\angle ADB = \angle ADC$, AD is perpendicular to BC (Def. 7); and since $\angle BAD = \angle CAD$, $\angle BAC$ is bisected.

Also the straight line which bisects the base of an isosceles triangle, is perpendicular to it (I. Prop. 8).

PROP. III.

If a perpendicular (AD) be drawn from the middle point (D) of any line (BC), the lines (BA, CA) drawn from the extremities of that line to any point (A) in the perpendicular are equal.

[Same Diagram.]

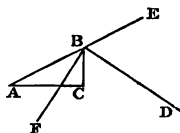
For since the triangles BAD, CAD have the sides $BD = CD$ (Hyp.), AD common, and $\angle ADB = \angle ADC$ (Def. 7), they shall have equal bases (I. Prop. 4); and therefore $AB = AC$.

If B and C be any points assumed in a line equidistant from D, and not the extremities of the line, still it is evident that the lines drawn from them to any one point in the perpendicular AD must be equal.

PROP. IV.

The lines (BF, BD) bisecting an internal angle and an external adjacent angle of a triangle (ABC), contain a right angle.

For $\angle$ s ABC, EBC, are together equal to two right angles (i. Prop. 13); therefore their halves $\angle$ s FBC, DBC (Hyp.) are together equal to half the sum of the whole angles, or to one right angle.

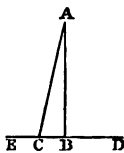


Hence it appears that if a line bisecting the angle A be drawn till it meets BD, it will make with the latter an angle equal to that made with BF by the line bisecting the external angle adjacent to A. For the angles in question are at the base of right-angled triangles, of which the other angles at the base are equal, being vertically opposite.

PROP. V.

If, from any point (A), two straight lines (AB, AC) be drawn to another straight line (ED), one of them (AB) perpendicular, the other not, the perpendicular shall fall at the side of the acute angle.

Since AB is perpendicular to ED, $\angle$ s ABC, ABD are equal (Def. 7); and since AC is not perpendicular to CD, $\angle$ s ACB, ACE, which are equal to two right angles (Prop. 13), are not equal, but one of them must be obtuse and the other acute; now $\angle$ ACB, at the side of which the perpendicular AB falls, is the acute angle. For $\angle$ s ABC, ACB are together less than two right angles (i. Prop. 17); but $\angle$ ABC is a right



angle (Hyp.); therefore $\angle ACB$ is less than a right angle; and consequently the perpendicular falls at the side of the acute angle.

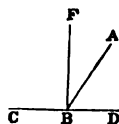
PROP. VI.

From the same point (A) to the same straight line (ED), there can be drawn but one perpendicular.

[See preceding Diagram.]

If AB be perpendicular to ED, $\angle ABC$ must be a right angle (Def. 7); and therefore $\angle ACB$ must be acute (i. Prop. 17); consequently, AC is not perpendicular to ED: and the same may be demonstrated of any other straight line, except AB, drawn from the point A to the line ED.

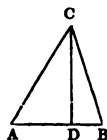
At the same point B in a straight line CD there can be but one perpendicular to that line. For if there were two perpendiculars BF, BA from the same point, there would be two right angles unequal, as one of them DBF would include the other DBA; which is absurd (Ax. 10).



PROP. VII.

The perpendicular (CD) is the shortest line which can be drawn from a given point to a given straight line (AB).

For since in the triangle ACD, CD subtends an acute angle, while CA subtends a right angle, CD is less than CA (i. Prop. 19). In like manner CD may be proved to be less than CB or any other line drawn from the point C to the line AB.



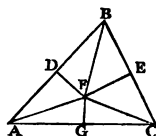
This proposition affords an obvious proof that two sides of any triangle ACB are greater than the third side.

For, a perpendicular CD being let fall on the third side, it is evident that AD will be less than AC , and DB than CB .

PROP. VIII.

If two sides (AB , BC) of a triangle (ABC) be bisected, and perpendiculars (DF , EF) be drawn from the middle points of those sides, till they meet, the lines (FA , FB , FC) drawn from the point of meeting of those perpendiculars to the angles of the triangle will be equal.

For since FE is perpendicular to, and bisects BC (Hyp.), $FB = FC$ (S. Prop. 8), and in like manner $FB = FA$; therefore also $FC = FA$ (Ax. 1).



Since $\angle FAB = \angle FBA$, and $\angle FCB = \angle FBC$ (I. Prop. 5), it follows that, if the angle ABC be a right angle, the point F must fall in the line AC , and if ABC be an obtuse angle, F must fall below AC .

PROP. IX.

If, from the same point of meeting (F) a perpendicular (FG) be let fall on the third side of the triangle, it will bisect it.

[See preceding Diagram.]

Since $FC = FA$ (S. Prop. 8); and also $\angle FAC = \angle FCA$ (I. Prop. 5), and $\angle FGA = \angle FGC$ (Hyp.), the triangles FAG , FCG have the sides $AG = GC$ (I. Prop. 26); and therefore the perpendicular FG bisects AC .

In like manner, if FG be drawn to the middle point of AC , it will be perpendicular to it (I. Prop. 8).

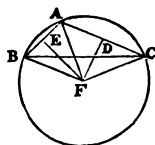
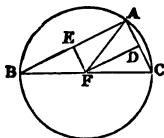
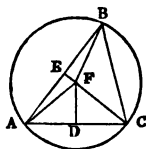
PROP. X.

The three perpendiculars (FD, FE, FG) drawn from the middle points of the three sides of a triangle, intersect in the same point (F).

[See preceding Diagram.]

If it be supposed that a line drawn perpendicular from the middle point of the side AC, does not pass through the point F, in which meet the perpendiculars drawn from the middle points of AB and BC: join FG, and then FG will be perpendicular to AC (s. Prop. 2); but the supposed line is also perpendicular to AC (Hyp.), and therefore from the same point G are drawn two straight lines, both perpendicular to AC, which is absurd (s. Prop. 6). Therefore the supposed line is not perpendicular to AC, and the same may be proved of any other line drawn from G except GF. Therefore the perpendiculars drawn from the middle points of the three sides of the triangle, pass through the same point.

This point, as is shown in the Fourth Book, is the centre of the circumscribed circle. Hence it follows, that there is but one circle capable of being circumscribed round a given triangle.

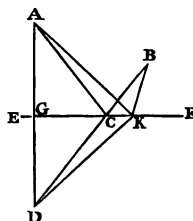


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PROP. XI.

Of any two lines which may be drawn from two given points (A, B) outside of a straight line (EF), to any point in it, those are together least which make equal angles with that line.

Let AC, BC make equal angles with EF; then if any point K be taken in EF, and AK, BK be drawn, these lines shall be together greater than AC, BC. For draw the perpendicular AG, and produce it till GD is equal to AG: join CD and KD. Now because $\angle ACG = \angle BCF$ (Hyp.), and also $\angle ACG = \angle DCG$ (I. Prop. 4), $\angle DCG = \angle BCF$ (Ax. 1); and since $\angle$ s GCB, BCF together equal two right angles, $\angle$ s GCB, DCG, also equal two right angles, and therefore DC and CB form one straight line (I. Prop. 14). But $AC = CD$ and $AK = KD$ (S. Prop. 3); therefore DB is equal to AC, BC taken together, and DK, BK are equal to AK, BK. But DK, KB are together greater than DB (I. Prop. 21), and therefore AK, BK are greater than AC, BC.



When a ray of light (AC) falling on a smooth surface (EF) is reflected by it, it makes the angle of reflection (BCF) equal to the angle of incidence (ACE). Hence it follows that reflected light takes the shortest possible course.

The construction of the above proposition readily suggests the solution of the following

PROBLEM.

To draw to the same point in a given indefinite straight line (EF), from two given points (A and B)

on the same side of that line, two straight lines making equal angles with it.

[See preceding Diagram.]

From A draw AG perpendicular to EF; produce AG till the produced part GD is equal to it; join BD, cutting EF in C, and join AC. The lines AC and BC, drawn from the given points to the same point in the given straight line, are the lines required. For the triangles AGC, DGC are equal (i. Prop. 4), and $\angle ACG = \angle DCG$; but $\angle DCG = \angle BCF$ (i. Prop. 15), and therefore $\angle ACG = \angle BCF$. The lines AC, BC therefore make equal angles with EF.

It must be observed that there cannot be drawn from A and B to any one point except C, in the line EF, lines making equal angles with it. For assume the point K in EF, and join AK, BK. Then $\angle BKF$ is greater than $\angle BCK$ (i. Prop. 17), and is therefore greater than $\angle ACE$; but $\angle ACE$ is greater than $\angle AKC$, and therefore $\angle BKF$ is greater than $\angle AKC$. The angles made by the lines drawn from the given points AB to K are therefore unequal, and the same may be proved with respect to any point but C.

PROP. XII.

If the line (AD) bisecting the vertical angle of a triangle (ABC) be perpendicular to its base (BC), the triangle must be isosceles.

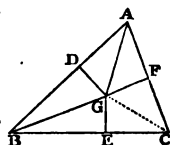
For the triangles ABD, ACD have respectively $\angle BAD = \angle CAD$ (Hyp.), $\angle ADB = \angle ADC$ (Hyp.), and the side AD lying between the equal angles common; they have therefore also $\angle ABD = \angle ACD$ (i. Prop. 26), and therefore the triangle BAC is isosceles.



PROP. XIII.

If lines (AG, BG) be drawn bisecting any two angles of a triangle (ABC), and from the point of their intersection (G) perpendiculars be let fall on the sides, these perpendiculars shall be equal.

The triangles BGD, BGE have $\angle DBG = \angle EBG$ (Hyp.), $\angle GDB = \angle GEB$ (Hyp.), and the side BG, opposite to equal angles, common. Therefore $GD = GE$ (I. Prop. 26); and in like manner it can be proved that $GD = GF$; therefore $GE = GF$.



It is manifest that if GC be drawn, it will bisect the angle ACB. For since $\angle GEC, GFC$ are right angles, $GE^2 + EC^2 = GC^2$ (I. Prop. 47) $= GF^2 + FC^2$; therefore $GE^2 + EC^2 = GF^2 + FC^2$. But $GE^2 = GF^2$, since $GE = GF$; therefore $EC^2 = FC^2$ and $EC = FC$. Consequently the triangles GEC, GFC are equal (I. Prop. 8), and $\angle ECG = \angle FCG$. Therefore GC bisects $\angle ACB$.

PROP. XIV.

The lines (AG, BG, CG) bisecting the three angles of a triangle (ABC), intersect in the same point.

[See preceding Diagram.]

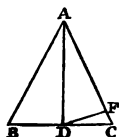
Let the lines AG, BG, respectively bisecting the angles BAC, ABC, meet in the point G. The line bisecting the angle ACB shall also pass through G. For, if any line bisecting ACB be supposed not to pass through G; join CG, and it will bisect $\angle ACB$ (8. Prop. 13); consequently, there will have been drawn two lines each bisecting $\angle ACB$, which is absurd; therefore, the line bisecting $\angle ACB$ must pass through G.

It appears from the Fourth Book (Prop. 4), that G is the centre of the inscribed circle. It therefore follows that there is but one circle capable of being inscribed in a given triangle.

PROP. XV.

If the line (AD) bisecting the vertical angle of a triangle (ABC) also bisects the base, the triangle must be isosceles.

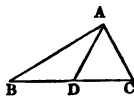
For if AB and AC be supposed not equal; let AC be the greater, and cut off from it AF equal to AB, and join DF. Then the triangles DAB, DAF shall be equal (I. Prop. 4), and $\angle ADB = \angle ADF$, and also $BD = DF$; but $BD = DC$ (Hyp.); therefore $DF = DC$, and $\angle DCF = \angle DFC$ (I. Prop. 5). Now, since $\angle BDF = \angle DCF + \angle DFC$ (I. Prop. 32), and these latter angles are equal; and likewise $\angle BDF = \angle ADB + \angle ADF$, and these latter angles are also equal; it follows, that $\angle ADB = \angle DCF$. But $\angle ADB$ is also greater than $\angle DCF$ (I. Prop. 16); which is absurd. Therefore, AB and AC are not unequal; and the triangle BAC is isosceles.



PROP. XVI.

In a right-angled triangle (ABC), the line (AD) drawn from the vertex of the right angle to the middle point of the base, is equal to half of the base.

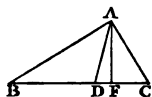
For if AD were greater than DC, then $\angle ACD$ would be greater than $\angle DAC$ (I. Prop. 18); and, in like manner, $\angle ABC$ would be greater than $\angle DAB$; and therefore, the angles at B and C would be together greater than the angle at A, which is absurd (I. Prop. 32, cor. 2). And if AD were less than BD or DC, it would follow, in like manner, that $\angle s$ B and C would be less than $\angle A$, which is also absurd. AD is therefore equal to BD and to DC.



PROP. XVII.

In a right-angled triangle (ABC), the angle (DAF) contained by the line bisecting the right angle, and the perpendicular drawn from the vertex of the right angle to the base, is equal to half of the difference of the angles at the base.

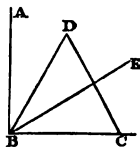
For $\angle DAC = \angle FAC + \angle DAF$; and if $\angle DAF$ be added to both, $\angle DAC + \angle DAF = \angle FAC + 2\angle DAF$. But since $\angle DAC = \angle DAB$ (Hyp.), $\angle DAC + \angle DAF = \angle BAF$; and therefore $\angle BAF = \angle FAC + 2\angle DAF$. But $\angle BAF = \angle C$, and $\angle FAC = \angle B$ (I. Prop. 32, Cor. 1); and therefore $\angle C = \angle B + 2\angle DAF$, or $\angle DAF$ is half of the difference of $\angle$ s C and B.



PROP. XVIII.

To trisect a right angle (ABC), or to divide it into three equal angles.

Let ABC be the given right angle. On one of its legs, BC, assume, from the vertex, a portion BD, and on it construct an equilateral triangle, BDC. Draw BE bisecting $\angle DBC$. The right angle ABC will be then trisected. For $\angle DBC$ is equal to two-thirds of a right angle (I. Prop. 32, Cor. 3); and therefore $\angle ABD$ is one-third of a right angle; and also $\angle DBE$, $\angle EBC$, being halves of $\angle DBC$, are thirds of a right angle. Therefore $\angle ABC$ is trisected.

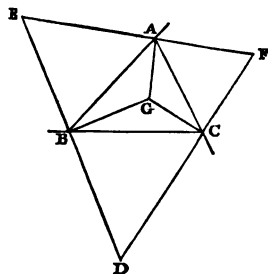


PROP. XIX.

If the external angles of a triangle (ABC) be bisected, the differences of the angles of the triangle (DEF) formed by the bisecting lines will be respectively one-half of the differences of the angles of the triangle, of which the external angles are bisected.

Draw AG , BG , CG bisecting the internal angles of the triangle; they shall meet in the same point, G (s. Prop. 14).

Since the angles of the quadrilateral figure, $DBGC$, are equal to four right angles (I. Prop. 32, Cor. 4), and $\angle s$ DBG , DCG are right angles (s. Prop. 4), $\angle s$ D , BGC are together equal to two right angles. But the three angles of the triangle BGC also equal two right angles (I. Prop. 32); therefore $\angle D = \angle GBC + \angle GCB$. In like manner it can be proved that



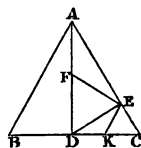
$\angle E = \angle GBA + \angle GAB$. But since $\angle GBC = \angle GBA$ (Const.), the difference of $\angle s$ D , E is equal to the difference of $\angle s$ GCB , GAB , that is to say, to the difference of half of the $\angle s$ BAC , ACB , or to half of the difference of these angles. In like manner it may be shown, that the differences of $\angle s$ E and F , and of $\angle s$ D and F , are respectively half of the differences of $\angle s$ ABC and ACB , and of $\angle s$ CBA and CAB .

PROP. XX.

If in an equilateral triangle (ABC), a perpendicular (AD) be drawn from one of the angles to the opposite side, and from the point (D) where it meets that

side, a perpendicular (DE) be drawn to another side, it will cut off a fourth part of the latter side; and if, from the point (E) where it falls, a line (EF) be drawn to the middle point of the first drawn perpendicular, this line, with the segment of the one perpendicular, and the whole of the other, will form an equilateral triangle (DEF).

Bisect DC in K , and join EK . Then, since DEC is a right-angled triangle, $EK = KC$ (s. Prop. 16), and the triangle CKE is therefore isosceles; consequently, $\angle KEC = \angle KCE$ (i. Prop. 5) = two-thirds of a right angle; and therefore $\angle EKC$ also equals two-thirds of a right angle (i. Prop. 32), and the triangle EKC is equilateral (i. Prop. 6, Cor.). Therefore, $EC = KC$, or half of DC , which is half of BC , and is therefore equal to one-fourth of BC or of AC .



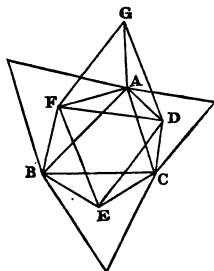
And since $\angle DEA$ is a right angle, and $\angle DAE$ a third of a right angle, $\angle ADE$ is equal to two-thirds of a right angle. But $EF = FD$ (s. Prop. 16), and therefore $\angle FED = \angle FDE$ (i. Prop. 5); consequently, $\angle DFE$ is also equal to two-thirds of a right angle (i. Prop. 32), and the triangle DFE is equilateral.

PROP. XXI.

If on the sides of any triangle (ABC) be constructed equilateral triangles, and in each of these triangles, lines (AF, BF ; AD, CD ; BE, CE) be drawn bisecting two of their angles and intersecting, the lines joining those points of intersection will form an equilateral triangle (DEF).

From the point D draw DG , making $\angle ADG = \angle CDE$ (i. Prop. 23); make $DG = DE$, and join AG and FG . Then

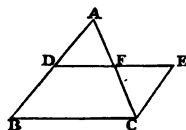
the triangles CDE, ADG are equal (i. Prop. 4), $AG = CE$, and $\angle DAG = \angle DCE$. But since in the figure ADCEBF, the angles at D, E, F, being each equal to four-thirds of a right angle, are together equal to four right angles, the remaining angles at A, C, B are also together equal to four right angles (i. Prop. 32, Cor. 4). Consequently, since $\angle DAG = \angle DCE$, $\angle GAF = \angle EBF$. But $AF = BF$ (i. Prop. 6), and also $AG = EC = EB$, and therefore the triangle $GAF = EBF$ (i. Prop. 4), $\angle FGA = \angle FEB$, and $GF = EF$. Therefore, the triangles FGD, FED are equal (i. Prop. 8), and $\angle FGD = \angle FED$. But $\angle FGD = \angle FEB + \angle CED$; therefore $\angle FED$ also equals the sum of these angles, and is, consequently, half of $\angle BEC$, and equal to two-thirds of a right angle. In like manner it may be proved, that $\angle s$ EFD, EDF are each equal to two-thirds of a right angle, and therefore the triangle EFD is equilateral (i. Prop. 6, Cor.).



PROP. XXII.

The straight line (DF) which bisects the sides of a triangle (ABC) is parallel to its base (BC).

Produce DF till the produced part, FE, equals DF, and join EC. Then the triangles AFD, CFE are equal (i. Prop. 4), $CE = AD$, and $\angle FCE = \angle FAD$; consequently, CE is parallel to AB (i. Prop. 27). But $CE = AD = DB$, and therefore DE is parallel to BC (i. Prop. 33).



DE is also equal to BC; and DF, therefore, being equal to FE (Const.), is equal to half of BC.

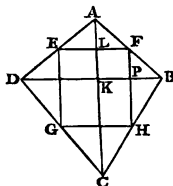
If DF be drawn from the middle point of AB , parallel to BC , it will bisect AC . For, let DF be produced till $DE=BC$; and, CE being drawn, the triangles AFD , CFE will be equal (i. Prop. 26), and therefore $AF=FC$.

It is evident that any line drawn from A to any point in BC will be bisected by DF .

PROP. XXIII.

If the sides of any quadrilateral figure ($ABCD$) be bisected, and their middle points joined, the figure ($EFHG$) thus inscribed, will be a parallelogram.

For draw the diagonals AC , BD ; and since EF bisects the sides of the triangle DAB , it is parallel to the base DB (s. Prop. 22). And for the same reason, GH is also parallel to DB ; therefore, GH is parallel to EF (i. Prop. 30). In like manner it may be proved, that EG is parallel to FH . The inscribed figure, $EFHG$, is therefore a parallelogram (Def. 26).



PROP. XXIV.

The parallelogram ($EFHG$) inscribed in a quadrilateral figure ($ABCD$), by joining the middle points of its sides, is equal to half of the quadrilateral figure.

[See preceding Diagram.]

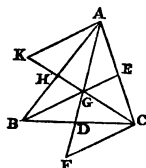
For the parallelogram KF is double of the triangle ALF , and also of the triangle FPB (i. Prop. 41); it is therefore equal to those triangles taken together, and is, consequently, half of the triangle AKB . In like manner, it is

evident, that the parallelograms KH, KG, KE are respectively halves of the triangles BKC, CKD, DKA; and therefore EFGH, which is composed of the parallelograms KF, KH, KG, KE, is half of the figure ABCD, which is composed of the triangles AKB, BKC, CKD, and DKA.

PROP. XXV.

In any triangle (ABC), if lines (AD, BE) be drawn from two of the angles so as to bisect the opposite sides, they shall intersect in such a manner that the segment of each towards the angle of the triangle shall be double of the segment towards the side.

Produce AD till it meets CF drawn parallel to BE. Then the triangles BDG, CDF, having the equal sides, BD, DC (Hyp.), lying between equal angles (i. Props. 15, 29), are equal (i. Prop. 26), and $GD = DF$. But $GF = AG$ (s. Prop. 22); and therefore AG is double of GD, which is half of GF.



If from the remaining angle (C) of the triangle, a line be drawn through the point of intersection (G) of those lines bisecting the two sides, it will bisect the third side.

Produce CG till it meets AK, drawn parallel to EB or CF. Then the triangles AGK, FGC, having the equal sides AG, FG lying between equal angles, are equal (i. Prop. 26); and therefore $AK = FC = BG$. The triangles AHK, BHG are therefore equal (i. Prop. 26), and $AH = BH$; consequently, AB is bisected in H.

PROP. XXVI.

The lines drawn from the angles of a triangle, to bisect the opposite sides, intersect in the same point.

[Preceding Diagram.]

If any line drawn from the angle C to AB , and not passing through G , be supposed to bisect AB ; draw CH passing through G , and it will also bisect AB (s. Prop. 25). AB is therefore bisected in two different points, which is absurd. Therefore no line drawn from C can bisect BA , but that which passes through G .

PROP. XXVII.

If from two angles (A, B) of a triangle (ABC) lines be drawn bisecting the opposite sides, and one of those lines (AD) be produced till it meets a line (CF) drawn parallel to the other, from the vertex of the angle made by the two bisected sides, the external part (DF) of the bisecting line, intercepted between the side of the triangle and the line last drawn, will be a fourth of the whole produced line (AF).

[Same Diagram.]

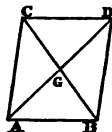
For the triangles GDB, FDC are equal (i. Prop. 26), and therefore $FD = DG$. FD is therefore half of FG , which is half of FA (s. Prop. 22); and consequently, FD is a fourth of FA .

If from E a line be drawn parallel to AF , and meeting BC , it will cut off a fourth part of BC ; or if drawn from D parallel to BE and meeting AC , it will cut off a fourth of AC . For, in the latter case, it must bisect GC (s. Prop. 22); and consequently also EC , which is half of AC .

PROP. XXVIII.

The diagonals (AD, BC) of a parallelogram (ACDB) bisect each other.

For the triangles AGB, CGD, having the equal sides AB, CD (i. Prop. 34) between equal angles (i. Prop. 29), are equal (i. Prop. 26); and therefore $AG = GD$, and $BG = GC$.



Any line drawn to the sides through the point of intersection of the diagonals will be bisected, and will bisect the parallelogram.

PROP. XXIX.

Every quadrilateral figure whose opposite angles or opposite sides are equal, is a parallelogram.

[Preceding Diagram.]

If the opposite angles of the quadrilateral figure ABDC be equal, then any adjacent two of those angles (for example, $\angle$ s CAB, ABD) are equal to half the sum of the four angles, or to two right angles; and consequently, CA and BD are parallel (i. Prop. 28). In like manner it may be inferred, that AB and CD are parallel. ABDC is therefore a parallelogram. If the opposite sides are equal, then the triangles ACD, ABD are equal (i. Prop. 8), and $\angle$ s BAD, ADC are equal; therefore AB and CD are parallel (i. Prop. 27); and consequently, AC and BD are also parallel (i. Prop. 33).

PROP. XXX.

The quadrilateral figure whose diagonals mutually bisect, is a parallelogram.

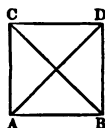
[Same Diagram.]

If the diagonals AD, CB mutually bisect, the triangles AGB, CGD are equal (I. Prop. 4); and therefore AB and CD are equal, and are also parallel (I. Prop. 27), since $\angle GBA = \angle GCD$. Therefore AC and BD are also parallel (I. Prop. 33); and therefore ABDC is a parallelogram.

PROP. XXXI.

The parallelogram whose diagonals are equal, is rectangular.

For the triangles BAC, ACD are equal (I. Prop. 8), and $\angle BAC = \angle ACD$; but these angles are together equal to two right angles (I. Prop. 29); therefore each of them is a right angle. Consequently, the figure is rectangular.



PROP. XXXII.

Parallelograms, the diagonals of which intersect at right angles, are equilateral.

For the adjacent sides subtend equal angles (Hyp.) at the intersection of the diagonals; which angles are contained by equal sides (S. Prop. 28). The adjacent sides are therefore equal (I. Prop. 4), and the parallelogram is equilateral (I. Prop. 34).

Conversely, if the parallelograms be rhombs, or equilateral, their diagonals shall intersect at right angles (8. Prop. 2).

PROP. XXXIII.

A quadrilateral figure bisected by both its diagonals, is a parallelogram.

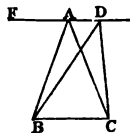
[Preceding Diagram.]

For the triangles ACB , ADB being both halves of the figure (Hyp.), are equal; and since they are on the same base, they must be between the same parallels (i. Prop. 39), or CD must be parallel to AB . In like manner, it may be proved, that AC is parallel to BD . The figure is therefore a parallelogram.

PROP. XXXIV.

Of all the triangles on the same base (BC), and between the same parallels (BC , FD), that has the least perimeter (or sum of the bounding lines) which has its sides (AB , AC) equal.

Through A draw FD parallel to BC ; and since $AB=AC$, $\angle C=\angle B$ (i. Prop. 5), and therefore since FD is parallel to BC , $\angle BAF=\angle CAD$ (i. Prop. 29); and consequently, BA and CA are together less than the sum of any other two lines drawn from B and C to any point in FD (8. Prop. 11). Therefore, the sides of the triangle BAC are together less than the sum of the sides of any other triangle, as BDC , standing on BC , and having its vertex in FD .



PROP. XXXV.

Of triangles having equal perimeters, the equilateral triangle is the greatest.

It is evident, from the preceding Proposition, that if a triangle has any two sides unequal, it is possible to construct a triangle equal to it, but having a less perimeter. The equilateral triangle, not having any two sides unequal, cannot be so reduced, and is therefore, of all the triangles of equal area, that which has the least perimeter. Consequently, of triangles having equal perimeters, the equilateral triangle has the greatest area.

PROP. XXXVI.

Of equal parallelograms on the same base, the rectangular is that which has the least perimeter.

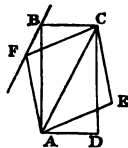
For the rectangular parallelogram has two sides equal to two sides of any other parallelogram on the same base with it; and its other two sides are less than the remaining sides of the other parallelogram on the same base and between the same parallels with it, since the perpendicular is less than any other line which can be drawn from a given point to a given straight line.

Hence it follows, that a rectangular parallelogram on the same base and between the same parallels as a rhomb, must be less than the square of that base, which square is therefore greater than the rhomb, and has an equal perimeter with it. Consequently, if a rhomb and a square be equal, the latter has a less perimeter than the former.

PROP. XXXVII.

Of equal parallelograms, that which has the least perimeter is the square.

Let $ABCD$ be a rectangular parallelogram, the sides of which are not equal. Draw the diagonal AC ; through B draw BF parallel to AC , and draw AF , CF , making equal angles with BF . These lines AF , CF make equal angles with AC (I. Prop. 29), and are therefore equal (I. Prop. 6); and they are together less than $AB + BC$ (S. Prop. 11). But the triangle AFC is equal to ABC (I. Prop. 37); and therefore if the rhomb $AFCE$ be completed, it will be equal to the parallelogram $ABCD$, but will have a less perimeter. And again, the rhomb $AFCE$ is less than a square of equal perimeter (S. Prop. 36). A square of equal area therefore has a less perimeter than $AFCE$, and therefore less also than $ABCD$.



PROP. XXXVIII.

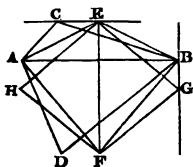
If an equilateral triangle and a square have equal perimeters, the triangle will be less than the square.

For the equilateral triangle is equal to a rectangular parallelogram on half of its base and between the same parallels (I. Prop. 41), but has a greater perimeter than it (S. Prop. 7); and that parallelogram has a greater perimeter than a square of equal area (S. Prop. 37). Therefore, if an equilateral triangle and a square be equal, the triangle must have the greater perimeter; and, consequently, if the perimeters be equal, the triangle must be less than the square.

PROP. XXXIX.

If a trapezium (ACBD) and a square have equal perimeters, the trapezium will be less than the square.

Through the angle C of the trapezium ACBD, draw CE parallel to the diagonal AB, and on AB construct the isosceles triangle AEB (s. Pr. 11); and in the same manner construct the isosceles triangle AFB equal to ADB. Then it is evident that the figure AEBF is equal to ACBD (I. Prop. 37), and has a less perimeter (s. Prop. 11). Again, draw EF, the diagonal of the figure AEBF, and it will bisect that figure (I. Pr. 8). On EF construct the isosceles triangles EGF, EHF equal to EBF, EAF, and therefore equal to each other. It is evident that EGFH is a rhomb equal to AEBF, but having a less perimeter. It is therefore equal to, but has a less perimeter than ACBD. But a square equal to the rhomb EGFH has a less perimeter than it; and therefore a less perimeter than the trapezium ACBD. Consequently, a square having an equal perimeter with the trapezium ACBD will be greater than it.



PROP. XL.

The sum of the three perpendiculars drawn to the sides of an equilateral triangle, from any point within it, is constant.

For if lines be drawn from the same point to the angles of the triangle, they will divide the equilateral triangle into three triangles, each of which is equal to the rectangular parallelogram, having for its adjacent sides half of the side of the equilateral triangle and the perpendi-

cular let fall on it from the assumed point (i. Prop. 41). Those three parallelograms, therefore, being equal to the equilateral triangle, are together equal to the rectangular parallelogram, having for its adjacent sides half of the side of the equilateral triangle and its altitude. The three perpendiculars are therefore always equal to the altitude of the triangle.

PROP. XLI.

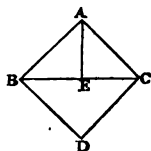
The square of a line is equal to four times the square of its half.

The parallelograms about the diagonal of a square are squares; for they are divided by the diagonal into equal triangles (i. Prop. 26); and have their adjacent sides equal: they are therefore equilateral; and being rectangular, they are also equiangular, and are therefore squares. If the parallelograms about the diagonal of a square be the squares of half the line, then the complements, having their adjacent sides equal, are also squares, and the square of the whole line is therefore equal to four times the square of its half.

PROP. XLII.

About a given line (BC) to place the square of which it is the diagonal.

Bisect the given line BC, and from the point of bisection E, draw EA perpendicular to BC, and make it equal to BE or EC; join BA, AC; and complete the parallelogram BACD. It is the square of which BC is the diagonal. For it is evident, that the triangles AEB, AEC are equal (i. Prop. 4); and $AB = AC$; and also, that $\angle s$ BAE, CAE



are each of them half of a right angle; and therefore $\angle BAC$ is a right angle. The parallelogram $BACD$ is consequently equiangular, and being also equilateral, it is a square.

PROP. XLIII.

If in a right-angled triangle the hypotenuse be equal to 5, and one of the sides equal to 3, what is the value of the other side?

Let x stand for the other side; then the square of the hypotenuse, or $25 = 9 + x^2$ (I. Prop. 47); if 9 be taken from both, then $25 - 9$ or $16 = x^2$. The other side, therefore, or x , is equal to the square root of 16, or to 4.

PROP. XLIV.

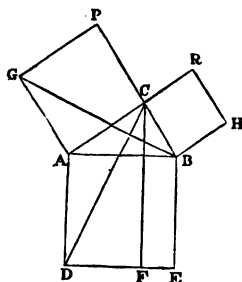
The square of a line bisecting an angle of an equilateral triangle, is equal to three times the square of the half of the side on which it falls.

The square of the side of an equilateral triangle is equal to the squares of the altitude and half of the side on which it falls (I. Prop. 47). But the square of the half side is a fourth of the square of the whole (S. Prop. 41); the square of the altitude, therefore, is the remaining three-fourths of the square of the whole side, and is consequently equal to three times the square of the half side.

PROP. XLV.

If, in the diagram of the Pythagorean Theorem, a line be drawn joining the angles H and E, G and D, the triangles HBE and GAD will be equal.

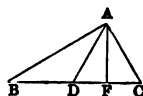
For since $HB = CB$, and $BE = AB$, and $\angle s$ HBE, CBA are together equal to two right angles, if BH be placed on BC, so as to coincide with it, AB and BE will form one continued straight line (i. Prop. 14), and therefore the triangles CBA, HBE are equal (i. Prop. 38). In like manner, the triangle GAD may be shown to be equal to CBA; and therefore HBE and GAD are equal.



PROP. XLVI.

If in a right-angled triangle (ABC) a perpendicular (AF) be let fall on the hypotenuse, from the right angle, the squares of the segments into which it divides the hypotenuse, will be equal to the square of the difference of those segments, together with twice the square of the perpendicular.

Bisect the hypotenuse BC in D, and join AD. Then, since $AD = BD$ (8. Prop. 16), $4 AD^2 = BC^2$ (8. Prop. 41) $= AB^2 + AC^2$ (i. Prop. 47). But $4 AD^2 = 4 AF^2 + 4 DF^2$ (i. Prop. 47); and therefore, $AB^2 + AC^2 = 4 AF^2 + 4 DF^2$. Take from both sides $2 AF^2$, and then



$BF^2 + FC^2 = 2 AF^2 + 4 DF^2$. But DF is half of the difference of the segments, and therefore $4 DF^2$ is equal to the square of that difference (s. Prop. 41). Consequently, the squares of the segments BF , FC are together equal to twice the square of the perpendicular AF , and the square of the difference of the segments.

PROP. XLVII.

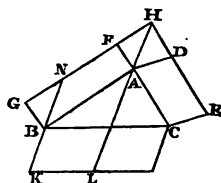
In a rhomb the squares of all the sides are together equal to the squares of the diagonals.

For in a rhomb the diagonals intersect at right angles (s. Prop. 32), and mutually bisect (s. Prop. 26). Therefore, the squares of the sides of the rhomb are equal to four times the squares of the segments of the diagonals (1. Prop. 47), and are, consequently, equal to the squares of the diagonals (s. Prop. 41).

PROP. XLVIII.

If on two sides (BA , CA) of a triangle (ABC) any parallelograms be described, and the remote sides (GF , ED) of those parallelograms be produced till they meet, and if a line (HA) be drawn joining that point of meeting and the vertex of the angle on the legs of which were constructed the parallelograms, these parallelograms are together equal to a parallelogram on the third side (BC) of the triangle; the other side (BK) of which parallelogram is equal and parallel to that joining line (HA).

Produce HA to L , and KB to N . Then $BN = AH$ (i. Prop. 34), and therefore $BN = BK$, and the parallelograms BH and BL are equal (i. Prop. 36); but the parallelograms BH and BF are likewise equal (i. Prop. 35); therefore BL and BF are equal. In like manner the parallelograms LC and CD may be shown to be equal, and therefore the whole parallelogram KC is equal to the parallelograms BF and CD .



SECOND BOOK.

PROP. XLIX.

Given the hypotenuse and the sum of the sides of a right-angled triangle, to find the sides.

Take the square of the hypotenuse from the square of the sum of the sides; and again, take twice the remainder from the square of the sum of the sides: add the square root of the last remainder to the sum of the sides, and half of the result will be the greater side, which, deducted from the sum of the sides, will leave the less side.

For the square of the sum of the sides is equal to the squares of the sides together with twice the rectangle contained by the sides (II. Prop. 4). But the square of the hypotenuse being equal to the squares of the sides (I. Prop. 47), if it be taken from the square of the sum of the sides, the remainder will be equal to twice the rectangle under the sides; and if the double of this remainder, or four times the rectangle under the sides, be taken from the square of the sum of the sides, there will remain the square of the difference of the sides (II. Prop. 8, and App. C, p. 166). But the sum and the difference of the sides added together, are equal to twice the greater side (App. C, p. 165), which is thus found, and being deducted from the sum leaves the less.

PROP. L.

Given the area of a right-angled triangle and its altitude (or perpendicular from the vertex of the right angle to the hypotenuse), to find the sides.

Divide twice the area by the altitude, and add the square of the quotient to four times the area. The square root of this sum will be equal to the sum of the sides, from which, and the above-mentioned quotient, which is equal to the hypotenuse, the sides of the triangle may be found.

For the area of the triangle is equal to half of the rectangle under the hypotenuse and altitude (i. Prop. 41), and therefore twice that area divided by the altitude gives the hypotenuse. But the area is also equal to half of the rectangle under the sides of the triangle (i. Prop. 34); and therefore four times the area, or twice the rectangle under the sides, and the square of the hypotenuse, which is equal to the squares of the sides (i. Prop. 47), are together equal to the square of the sum of the sides (ii. Prop. 4). But the sum of the sides being known, and also the hypotenuse, the sides of the triangle may be found by the preceding Proposition.

PROP. LI.

Given the segments of the hypotenuse made by the perpendicular from the right angle, to find the sides.

Add the rectangle under the segments (or, in numbers, their product) to the square of each segment, and the square roots of the sums so obtained will be equal to the sides.

For the rectangle under the segments of the hypotenuse is equal to the square of the altitude (ii. Prop. 14), which

square, together with the square of either segment, is equal to the square of the side adjacent to that segment (i. Prop. 47).

Since the square of the hypotenuse is equal to the squares of its segments, together with twice the rectangle under the segments (ii. Prop. 4); and also to the squares of the sides, which are equal to the squares of the segments of the hypotenuse, together with twice the square of the altitude (i. Prop. 47), it follows, that the squares of the segments with twice the rectangle under them, are equal to the squares of the segments with twice the square of the altitude: and thus it may be concluded, without reference to the last Proposition of the Second Book, that the rectangle under the segments is equal to the square of the altitude.

PROP. LII.

Given the hypotenuse and the difference of the sides,
to find the sides.

From the square of the hypotenuse take the square of the given difference: to twice the remainder add the square of the difference, and the square root of the sum so obtained will be the sum of the sides.

For the square of the hypotenuse is equal to the squares of the sides (i. Prop. 47), or to twice the rectangle under the sides with the square of their difference (ii. Prop. 7, & App. C, p. 167). The latter square, therefore, being taken from the square of the hypotenuse, leaves twice the rectangle under the sides. But if to the double of this remainder, or four times the rectangle under the sides, be added the square of the difference, the sums thus obtained will be the square of the sum of the sides (ii. Prop. 8, and App. C, p. 166). From the sum of the sides thus found, and their difference, which is given, the sides themselves may be deduced (App. C, p. 165).

PROP. LIII.

Given the altitude and the difference of the segments of the hypotenuse, to find the sides.

To the square of the altitude add the square of half of the given difference. The square root of the sum thus obtained, together with half of the given difference, will be equal to the greater segment, which, by subtracting the difference, gives the less segment. From the segments of the hypotenuse and the altitude may be obtained the sides.

For the square of the altitude is equal to the rectangle under the segments (II. Prop. 13), which, with the square of half of the difference, is equal to the square of half of the hypotenuse (II. Prop. 5); and half of the hypotenuse, increased by half of the difference of its segments, is equal to the greater segment; or, diminished by that difference, to the less segment (App. C, p. 165). The segments of the hypotenuse being thus obtained, if to the square of each of them be added the square of the altitude, the result will be equal to the square of the conterminous side (I. Prop. 47).

PROP. LIV.

Given one side and the conterminous segment of the hypotenuse, to find the other side.

Divide the square of the given side by the given segment. The quotient will be equal to the hypotenuse, the square of which, diminished by the square of the given side, will be equal to the square of the other side.

For the square of a side of a right-angled triangle is equal to the rectangle under the hypotenuse, and the conterminous segment (I. Prop. 47). From the side, therefore, and segment, may be deduced the hypotenuse, the square

of which is equal to the squares of the sides ; and consequently, the difference between the square of the hypotenuse and that of the given side, is equal to the square of the other side.

PROP. LV.

Given the line drawn from the vertex of the right angle to the middle point of the hypotenuse and the difference of the sides, to find the sides.

The line drawn from the vertex of the right angle to the middle point of the hypotenuse is equal to half of the hypotenuse (s. Prop. 16). This Proposition, therefore, resolves itself into the 52nd. And in general, whenever the hypotenuse is among the data for determining right-angled triangles, the line drawn from the vertex of the right angle to the middle point of the hypotenuse may be substituted for it.

If, with the line from the vertex bisecting the hypotenuse, be also given the line drawn from the middle point of the preceding line perpendicular to the hypotenuse, or to one of the sides, then also the triangle can be determined. For in the former case, the perpendicular is equal to half of the altitude (s. Prop. 22) ; and when it falls on one of the sides, it is equal to a fourth of the other side. But the hypotenuse and altitude being known, the sides may be readily found ; for the square of the altitude is equal to the rectangle under the segments. Four times the square of the altitude, therefore, deducted from the square of the hypotenuse, leaves the square of the difference of the segments (ii. Prop. 8). The segments may be found, their sum and difference being thus known.

PROP. LVI.*

Given one side, and the difference between the other and the hypotenuse, to determine the triangle.

Take the given difference from the given side: divide half the square of the remainder by the given difference, and the quotient added to the given side will give the hypotenuse; from which and the given side the other side may be found.

For, if parts equal to the sides be assumed on the hypotenuse, it will then be divided into $A + B + C$; and if the side $A + B$ be given, and also A , which is the difference between the side $C + B$ and the hypotenuse, the latter may be found, by taking A from $A + B$, when B will remain, the square of which is equal to $2 AC$ (s. Prop. 62).^{*} Dividing AC , therefore, by A , C is found, which, added to $A + B$, completes the hypotenuse. The square of the given side, taken from the square of the hypotenuse, leaves the square of the other side.

PROP. LVII.

If the hypotenuse be 5, and one side 4, what is the numerical value of the other side?

Since the squares of the sides are equal to the square of the hypotenuse, or to 25, if 16, the square of the given side, be taken from 25, the remainder, 9, will be the square of the other side; which side is, consequently, 3.

In order to represent a right-angled triangle in whole numbers, it is necessary that the sum of the squares of the sides should be the square of a whole number. This is the case in the example adduced above, where 9 and 16 (the squares of 3 and 4 respectively) added together make 25, or the square of 5.

^{*} This Proposition has been inadvertently misplaced in the Appendix to the Elements. It ought to follow the 62nd, from which it is immediately derived.

PROP. LVIII.

If the sides be 4 and 3, what are the segments of the hypotenuse made by the altitude?

The sides being 4 and 3, the squares of which are 16 and 9, the hypotenuse must be the square root of the sum of their squares, or 5. But the square of each side is equal to the rectangle under the hypotenuse and the conterminous segment. If 16, therefore, the square of the greater side, be divided by 5, or the hypotenuse, the quotient $3\frac{1}{5}$ will represent the segment conterminous with the greater side. The remainder of the hypotenuse, or $1\frac{4}{5}$, will be the segment conterminous with the less side.

PROP. LIX.

The product of the parts of any number is less, the more unequally that number is divided.

If the parts be equal, their product is the square of half of their sum. But if they be not equal, then their product, together with the square of half their difference, is equal to the square of half their sum (li. Prop. 5). Their product, therefore, diminishes as their difference increases, or as they are more unequal.

PROP. LX.

If the squares of the numbers 1, 2, 3, 4, 5, &c. be deducted, each preceding from the succeeding square, the remainders will be the series of odd numbers, 1, 3, 5, 7, 9, &c.

The difference between the squares of two numbers is equal to the product of their sum and difference (App. C, p. 165), or to their sum, when their difference is only 1. Consequently, the difference of the squares of 1 and 2, will be 3×1 , or 3; that of the squares of 2 and 3, will be 5×1 , or 5; and in like manner, the difference of the squares of 3 and 4 will be equal to the sum $3 + 4$, or 7. Now, since these differences are the sums of the consecutive numbers, viz. $1 + 2$, $2 + 3$, $3 + 4$, &c., and consist therefore each of an odd and an even number, they must be odd numbers; and since every two such consecutive sums, as $1 + 2$, $2 + 3$, have one term in common, and therefore differ by the difference of their extreme terms, which is 2, they form the series of odd numbers.

PROP. LXI.

If the separate differences between the first of three magnitudes and the other two be added to or subtracted from the difference between it and the sum of those two, according as the first magnitude is greater or less than the others, the result will be equal to that first magnitude.

The differences between the first magnitude and the other two separately, or $A - B$ and $A - C$, are together equal to the difference between twice the first and the sum of the other two, or $2A - B - C$. If, therefore, the difference between the first magnitude and the sum of the other two, or $A - B - C$, be taken from the sum of the separate differences between the first and the other two, the remainder will be the first magnitude, or A .

PROP. LXII.

The square of the difference between the sum of the sides of a right-angled triangle and the hypotenuse, is equal to twice the rectangle under the differences of the sides and hypotenuse.

From the extremities of the hypotenuse, let portions be taken on it equal to the sides. It will then be divided into three parts, A, B, and C; $A + B$ and $C + B$ being respectively equal to the sides of the triangle. But the square of the whole hypotenuse is equal to the squares of its parts and twice the rectangles under its parts, or to $A^2 + B^2 + C^2 + 2 AB + 2 AC + 2 BC$ (App. C, p. 163): and it is also equal to the squares of $A + B$ and $C + B$, since these segments are equal to the sides of the triangle. But the square of $A + B = A^2 + 2 AB + B^2$, and the square of $C + B = C^2 + 2 CB + B^2$ (II. Prop. 4). These, added together, make $A^2 + 2 B^2 + C^2 + 2 AB + 2 CB$.

which is equal to $A^2 + B^2 + C^2 + 2 AB + 2 CB + 2 AC$. Now, if from these equal sums be taken all the common terms, the remainders must be equal (Ax. 3), and therefore $B^2 = 2 AC$. But since A, B, and C are equal to the hypotenuse, while $A + B$ and $C + B$ represent the sides, it is evident, that B is the difference between the sum of the sides and the hypotenuse, and that A and C are the separate differences of the hypotenuse and the sides. Therefore, &c.

Since the altitude of a right-angled triangle, or the line drawn from the vertex of the right angle perpendicular to the hypotenuse, divides the triangle into two right-angled triangles, which may be in like manner divided and subdivided; and as all the right-angled triangles resulting from such division may be determined from the data specified in the foregoing Propositions, and being determined, easily furnish the solution of the original triangle, it is obvious, that the data for the solution of a right-angled

triangle may be varied without end. Thus, for an example of data obtained by subdividing a triangle, suppose that from the point in which the altitude meets the hypotenuse, a perpendicular be drawn to one of the sides; and from the point where that perpendicular meets the side, a perpendicular be drawn to the hypotenuse. Now, either of the perpendiculars to the hypotenuse being known, and also the segment of the hypotenuse between them, the triangle may be determined.

PROP. LXIII.

If in an isosceles triangle (BAC), a line (AD) be drawn from the vertex to the base, twice the square of that line, with the squares of the segments (BD, DC) of the base, will be equal to the squares of the equal sides with the square of the difference of the segments of the base.

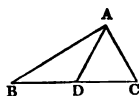
Draw the perpendicular AE, and it will bisect the base (s. Prop. 2). Then, since $\angle ADE$ is an acute angle (s. Prop. 5), the squares of AD and DC exceed the square of AC by twice the rectangle CD·DE (ii. Prop. 13), or $AD^2 + DC^2 = AC^2 + 2 CD \cdot DE$. And because $\angle ADB$ is an obtuse angle (i. Prop. 13), $AD^2 + BD^2 = AB^2 - 2 BD \cdot DE$ (ii. Prop. 12). Therefore, by addition, $2 AD^2 + BD^2 + DC^2 = AC^2 + AB^2 + 2 DC \cdot DE - 2 BD \cdot DE$; that is to say, twice the square of the line drawn from the vertex to the base, with the squares of the segments of the base, is equal to the squares of the equal sides, together with the excess of $2 DC \cdot DE$ above $2 BD \cdot DE$. But since BC is bisected in E, DE is half of the difference of the segments, or $DC - BD = 2 DE$; and therefore $2 DC \cdot DE - 2 BD \cdot DE$, or $2 DE \cdot (DC - BD) = 2 DE \cdot 2 DE = 4 DE^2$, or four times the square of half of the difference of the segments of the base, which is equal to the square of the whole difference.



PROP. LXIV.

If from any angle of a triangle (BAC) a line (AD) be drawn bisecting the opposite side, the squares of the sides containing that angle will be equal to twice the square of the bisecting line and twice the square of half of the bisected side.

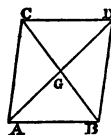
For, since the segments BD, DC of the base are equal, the square of AB, which subtends the obtuse angle, exceeds the squares of AD and BD (or DC) by the same double rectangle (ii. Prop. 12), by which the square of AC subtending the acute angle falls short of those squares (ii. Prop. 13). The squares of AB and AC are therefore equal to twice the square of AD, with the squares of the equal segments.



PROP. LXV.

In a parallelogram (ACDB), the sum of the squares of the sides is equal to the sum of the squares of the diagonals.

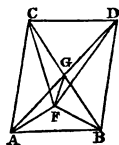
Since the diagonals of a parallelogram bisect each other (s. Prop. 28), the squares of AC and CD are equal to twice the squares of CG and AG (s. Prop. 64); and, in like manner, the squares of AB and BD are equal to twice the squares of BG and AG. But $BG = CG$, and therefore the squares of the sides of a parallelogram are equal to four times the squares of the halves of its diagonals; that is to say, to the squares of its diagonals.



PROP. LXVI.

In a parallelogram (ACDB), the sum of the squares of the lines drawn from its angles to any point (F) within it, is equal to half of the sum of the squares of the sides, together with four times the square of the line (FG) drawn from that point to the intersection of its diagonals.

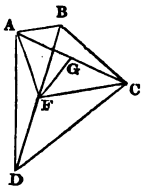
Since AD is bisected in G (s. Prop. 28), the squares of AF and FD are equal to twice the squares of FG and AG (s. Prop. 64). In like manner, the squares of BF and FC are equal to twice the squares of FG and GC. Therefore, the squares of AF, FD, BF, FC, taken together, are equal to twice the squares of AG and GC; that is, to the squares of AC and CD (s. Prop. 64), together with four times the square of FG.



PROP. LXVII.

In a trapezium (ABCD), the sum of the squares of the sides is equal to the sum of the squares of the diagonals together with four times the square of the line (FG) joining their middle points.

The squares of AF and CF are equal to twice the squares of FG and AG (s. Prop. 64), and therefore twice the squares of AF and CF are equal to four times the square of FG and four times the square of AG, that is, the square of AC. But the squares of DA and AB are together equal to twice the squares of AF and DF (s. Prop. 64); and, in like manner, the squares of DC and CB are equal to twice the squares of CF and DF. Therefore, the squares of DA, AB, DC, CB are



together equal to four times the square of DF, that is, to the square of DB, with twice the squares of AF and CF, which have been shown to be equal to the square of AC, together with four times the square of FG.

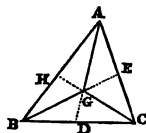
PROP. LXVIII.

If from the angles of any triangle lines be drawn bisecting the opposite sides, four times the squares of these lines will be equal to three times the squares of the sides of the triangle.

The squares of two sides of a triangle are equal to twice the square of the line drawn from the angle contained by them to the middle point of the opposite side, together with half of the square of that side (S. Prop. 64). Therefore, twice the squares of all the sides is equal to twice the squares of the three bisecting lines, with half of the squares of the sides; and consequently, four times the squares of the sides are equal to four times the squares of the bisecting lines, together with the squares of the sides. The common term being subtracted from both, four times the squares of the bisecting lines will remain equal to three times the squares of the sides.

It has been shown (S. Prop. 25), that when lines are drawn from the angles of a triangle to bisect the opposite sides, their segments intercepted between the point of their intersection and the angles, are each respectively equal to two-thirds of the whole line: and also that they intersect in the same point (S. Prop. 26). If the whole bisecting line, therefore, be represented by 3, its segment towards the angle will be 2; and their squares respectively will be 9 and 4. Therefore, four times the squares of the bisecting lines will be equal to nine times the squares of their segments towards the angles. But three times the squares of the sides equals four times the squares of the bisecting lines, and therefore equals nine times the squares of their

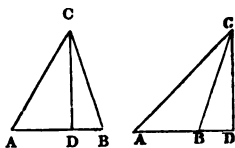
segments towards the angles. Consequently, these equals being divided by three, the squares of AB , AC , BC , the sides of any triangle ABC , are together equal to three times the squares of AG , BG , CG , the segments of the lines bisecting the sides, intercepted between the point of intersection of those lines and the angles of the triangle.



PROP. LXIX.

Given the sides of any triangle, to find its area.

Let the sides be AC , AB , and CB . Take the square of AC from the sum of the squares of AB and CB . Divide half of the remainder by one of the latter sides (AB), and take the square of the quotient from the square of the other (CB); the square root of the remainder, multiplied by AB , will give the area of the triangle.



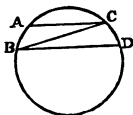
For half of the difference between the square of one side and the sum of the squares of the other two, is equal to the rectangle under one of the latter and the line intercepted between the angle contained by them and the perpendicular let fall from the angle opposite to that side (ii. Props. 12, 13). That side therefore being known, the intercepted line is also known; and its square being taken from the square of the other side, leaves the square of the perpendicular, the product of which, and half of the side on which it falls, gives the area of the triangle (i. Prop. 41).

 THIRD AND FOURTH BOOKS.

PROP. LXX.

Lines joining the adjacent extremities of equal arches (AB, CD) are parallel; and parallel chords (AC, BD) cut off equal arches.

Since the arches AB, CD are equal, the angles ACB, CBD standing on them at the circumference are equal (iii. Prop. 27). But these are alternate angles, and therefore the chords AC, BD are parallel (i. Prop. 27).



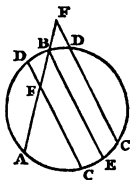
And conversely, if the chords AC, BD be supposed to be parallel, then $\angle$ s ACB, CBD are equal (i. Prop. 29), and therefore the arches on which they stand are equal (iii. Prop. 26).

 PROP. LXXI.

Two chords (AB, CD) intersecting, contain an angle equal to the sum or difference of the angles at the circumference, standing on the intercepted arches (AC, BD), according as the intersection is within or without the circle.

From the point B, in which one of the chords (AB) meets the circumference, draw BE parallel to the other chord, CD. Then, $\angle$ ABE is equal to $\angle$ AFC (i. Prop. 29).

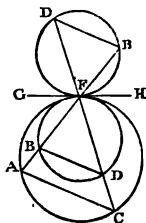
And because BE is parallel to DC, the arch BD is equal to the arch EC (s. Prop. 70). The arch AE, therefore, on which stands the angle ABE, is equal to the sum of the arches intercepted by the chords AB, CD, when they intersect within the circle; and to the difference of those arches, when the chords meet without. Consequently, $\angle AFC$, being equal to $\angle ABE$, is equal to the sum or difference of the angles standing on the intercepted arches, according as the meeting of the chords is within or without the circle.



PROP. LXXII.

If two circles (FAC, FBD) touch, and from the point of contact (F) two chords be drawn meeting their circumferences, the chords (AC, BD) of the intercepted arches will be parallel.

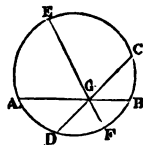
Through the point of contact (F) of the two circles, draw the straight line (GH) touching one of them, DFB (iii. Prop. 17). And since, if a line be drawn joining the centres of the circles, it must pass through the point of contact (iii. Props. 11, 12), and GH makes right angles with that line (iii. Prop. 18), it will be a tangent to the circle AFC also (iii. Prop. 16). But $\angle GFD$, made by the tangent GH and secant DC, is equal to $\angle FBD$, in the alternate segment of the circle FBD (iii. Prop. 32); and it is also equal to $\angle HFC$ (i. Prop. 15); therefore $\angle FBD = \angle HFC$ (Ax. 1). But $\angle HFC$ is equal to $\angle FAC$, in the alternate segment of the circle FAC (iii. Prop. 32); and therefore $\angle FBD = \angle FAC$. But these are alternate angles, and consequently DB and AC are parallel (i. Prop. 27).



PROP. LXXIII.

If two chords (AB, CD) intersect, the difference of their squares is equal to the difference of the squares of the differences of their segments.

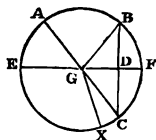
If the chord AB be bisected, then the rectangle under the unequal segments AG, GB, with the square of half their difference, will be equal to the square of half of AB (II. Prop. 5). And, in like manner, the chord CD being bisected, the rectangle under its unequal segments CG, GD, together with the square of half their difference, is equal to the square of half of CD. But the rectangles $AG \cdot GB$, $CG \cdot GD$, under the segments of the intersecting chords, are equal (III. Prop. 35). Therefore, the square of half of AB, minus the square of half of the difference of its segments, is equal to the square of half of CD, minus the square of half of the difference of its segments: or multiplying these equals by four, the square of AB, minus the square of the difference of its segments, is equal to the square of CD, minus the square of the difference of its segments. Let M represent the former difference, N the latter; then $AB^2 - M^2 = CD^2 - N^2$; and if M^2 be added to, CD^2 subtracted from both sides, $AB^2 - CD^2 = M^2 - N^2$, or the difference of the squares of AB and CD is equal to the difference of the squares of the differences of their segments.



PROP. LXXIV.

If equal arches (EA, FB) be assumed from the extremities of the diameter of a circle, and lines be drawn from the extremity of the one arch to the centre, from the other, perpendicular to the diameter, these lines (AG, BD) if produced will meet at the same point (c) in the circumference.

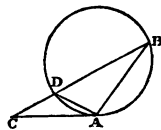
For if it be supposed that BD produced, meets the circumference in C, while AG produced, meets it in X; join GB and GC; and since $GB=GC$, $\angle GBC=\angle GCB$ (i. Prop. 5), and also $\angle GDB=\angle GDC$ (Hyp.), and therefore the triangles GBD, GCD are equal (i. Prop. 26), and $\angle BGD=\angle CGD$. But since the arches BF, AE are equal, the angles BGF, AGE standing on them are also equal (iii. Prop. 27), and therefore also $\angle FGC$ is equal to $\angle AGE$ (Ax. 1). But $\angle s$ AGE, AGF are together equal to two right angles (i. Prop. 13); and therefore $\angle s$ AGF, FGC also are together equal to two right angles, and consequently AG and GC form one straight line (i. Prop. 14). But it has been supposed, that AG produced would meet the circumference in X; therefore, both GC and GX are continuations of AG, which is absurd. AG produced, therefore, cannot meet the circumference in X, nor in any point but C.



PROP. LXXV.

If from one extremity of a chord (AB), a tangent (AC) be drawn to the circle, equal to that chord, and a line (CB) be drawn joining the remote extremities of the chord and tangent, the arch intercepted between that line and the tangent equals half of the arch subtended by the chord.

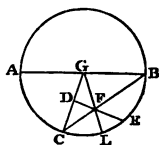
Join DA: and since AC is a tangent to the circle, and DA a secant, $\angle DAC$ is equal to $\angle ABD$ in the alternate segment (iii. Prop. 32); and because $AC=AB$ (Const.), $\angle ACB=\angle ABC$ (i. Prop. 5), and therefore $\angle s$ DAC, DCA are equal (Ax. 1). But $\angle ADB$ is equal to $\angle s$ DAC, DCA taken together (i. Prop. 32); it is therefore double of $\angle DCA$, and consequently it is also double of $\angle DBA$. The arch AB, on which it stands, is therefore double of the arch AD.



PROP. LXXVI.

If a radius (GC) be drawn making any angle with the diameter (AB) of a circle, and a perpendicular (DE) be drawn from its middle point to the circumference, and if a chord (CB) be drawn from the extremity of the radius to that of the diameter, so as to intersect the perpendicular, the radius (GL) drawn through that point of intersection (F) will divide the semi-circumference into two parts, respectively trisected by the first-drawn radius and the perpendicular to it.

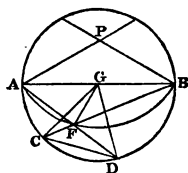
Since GC is bisected in D, and DE is perpendicular to GC, the triangles GDF and CDF are equal (i. Prop. 4), and $\angle FGD = \angle FCD$. But $\angle FCG$ or $FCD = \angle GBC$ (i. Prop. 5); therefore $\angle FGD$ or $\angle LGC = \angle GBC$. But $\angle AGC$ at the centre is double of $\angle GBC$ at the circumference (iii. Prop. 20), and therefore $\angle AGC$ is double also of $\angle LGC$. Therefore the arch AC is double of the arch CL, or CL is a third of AL. And if CE and GE be joined, it is manifest, from the equality of the triangles GDE, CDE (i. Prop. 4), that $CE = GE$; but also $GE = GC$; therefore GCE is an equilateral triangle, and the angle CGE, being a third of two right angles, is a third of the sum of $\angle$ s AGC, CGE, EGB, or half of the sum of $\angle$ s AGC, EGB. But it has been shown, that $\angle CGL$ is half of $\angle AGC$, and therefore $\angle LGE$ also is half of $\angle EGB$, and the arch LE therefore is half of the arch EB, or a third of LB.



PROP. LXXVII.

If from the extremities of the diameter (AB) of a circle, chords be drawn making with the diameter angles equal to one-third of a right angle, and from the point of intersection (P) of those chords, as centre, an arch (AFB) be described, of which the diameter of the circle is the chord, the two chords (AD, BC) drawn from the extremities of the diameter through any point (F) in that arch, will intercept an arch of the circle of which the chord (CD) is equal to the radius.

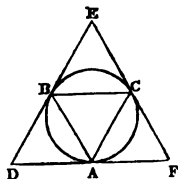
Since the $\angle$ s PAB, PBA are each equal to one-third of a right angle, $\angle$ APB is equal to four-thirds of a right angle (i. Prop. 32); and since it is at the centre, the angle at the circumference standing on the same arch AFB, must be equal to two-thirds of a right angle (iii. Prop. 20): and consequently, the angle complementary to it, or $\angle$ AFB, is equal to four-thirds of a right angle (iii. Prop. 22): therefore, $\angle$ s FAB, FBA are together equal to two-thirds of a right angle (i. Prop. 32). But if the lines GF, GC, and GD be drawn, then, since $GC = GB$, $\angle$ GCB = $\angle$ GBC (i. Prop. 5): and also $\angle$ DCB = $\angle$ DAB, because they stand on the same arch DB (iii. Prop. 27). Therefore, $\angle$ s GCB, DCB together equal $\angle$ s FAB, FBA together, or the angle GCD is equal to two-thirds of a right angle; and since $GC = GD$, $\angle$ GDC is also equal to two-thirds of a right angle, and therefore the triangle GCD is equilateral, and CD is consequently equal to the radius.



PROP. LXXVIII.

The equilateral triangle (ABC) which is inscribed in a circle, is equal to one-fourth of that (DEF) which is circumscribed about it.

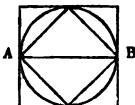
If lines AB , BC , CA be drawn, joining the points in which the circumscribed triangle DEF touches the circle, the triangle thus inscribed in the circle will be equilateral, and one-fourth of the triangle DEF . For since DA and DB are tangents to the circle from the same point, they are equal (III. Prop. 36), and since the angle at D is equal to two-thirds of a right angle (Hyp.), the equal angles DAB , DBA are also each equal to two-thirds of a right angle (I. Prop. 32), and therefore the triangle DAB is equilateral (I. Prop. 6). In like manner it may be shown that ACF and BEC are equilateral triangles; and since $\angle$ s DAB , FAC are each equal to two-thirds of a right angle, $\angle$ BAC also is equal to two-thirds of a right angle (I. Prop. 13); and also, it may be shown that $\angle$ s ABC , ACB are each equal to two-thirds of a right angle. Therefore ABC is an equilateral triangle (I. Prop. 6), and since it is equal to each of the triangles DAB , BEC , ACF (I. Prop. 26), it is a fourth of the circumscribed triangle DEF .



PROP. LXXIX.

The inscribed square is half of the circumscribed.

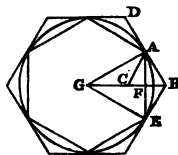
For the square of the diagonal AB of the inscribed square is double of the square of its side (I. Prop. 47); that is to say, it is double of the inscribed square. But the side of the circumscribed square is equal to that diagonal (I. Prop. 34), and therefore the circumscribed square is double of that inscribed.



PROP. LXXX.

The hexagon inscribed in a circle is three-fourths of that circumscribed.

Let the circumscribed hexagon touch the circle at the angles of the inscribed figure. Draw GB to any angle of the circumscribed hexagon, and also GA and GE to its adjacent points of contact. Then from the equality of the triangles GAB, GEB (i. Prop. 8), it is evident that $\angle AGB = \angle EGB$, and therefore the triangles



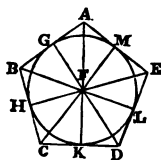
GAF, GEF are equal (i. Prop. 4), and $\angle s$ GFA, GFE are right angles. But $GB = GD = BD$ (iv. Prop. 15), and since GA is perpendicular to BD (iii. Prop. 18), it bisects it (s. Prop. 2), and therefore, in the triangle GAB, the line AC drawn from the vertex of the right angle to the middle point of the hypotenuse, being equal to half of the hypotenuse or to CB (s. Prop. 16), is equal to AB also, and the triangle ACB is equilateral; its base CB therefore is bisected by the perpendicular AF (s. Prop. 2). But since $GC = CB$, the triangle GAC = CAB (i. Prop. 38). And also since $CF = FB$, the triangle CAF = FAB; that is to say, CAB is half of GAB, and FAB one-fourth of GAB. Consequently the triangle GAF is three-fourths of GAB. In like manner if lines be drawn to all the angles of the inscribed and circumscribed hexagons, they will divide the figures into triangles, of which those composing the inscribed hexagon will be each respectively three-fourths of those composing the circumscribed.

PROP. LXXXI.

The area of any regular figure is equal to the rectangle under its semiperimeter (or half of the sum of its sides) and the radius of the circle inscribed in it.

For every regular figure is capable of being divided

into as many equal triangles as it has sides, by lines drawn to its angles from the centre of the inscribed circle; and the radius drawn to any of these sides is perpendicular to it (iii. Prop. 18). But each of these triangles is equal to a rectangle under its altitude, that is, the radius, and the half of its base, or of the side of the figure (i. Prop. 41); and therefore all the triangles together are equal to a rectangle under the radius of the inscribed circle and the half of the sum of the sides, or the semiperimeter; but all the triangles taken together compose the figure, the area of which is therefore equal to a rectangle under its semiperimeter and the radius of the inscribed circle.



PROP. LXXXII.

The three perpendiculars (AE , BG , CD) from the angles of any triangle (ABC) to the opposite sides, intersect in the same point.

LEMMA.*—If in any triangle, perpendiculars (BG , CD) be drawn from two of the angles (B and C) to the opposite sides, the line drawn from the third angle (A) to the opposite side through (F) the point of intersection of those perpendiculars, will be perpendicular to that side (BC).

For, since the $\angle$ s ADF , AGF are both right angles, the quadrilateral figure $ADFG$ is capable of being inscribed in a circle (iii. Prop. 22), and then $\angle DAF = \angle DGF$ (iii. Prop. 21). And again, since BDC and BGC are both right-angled triangles, and stand on the same base, they may be both inscribed in the same semicircle (iii. Prop. 31), and therefore

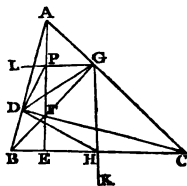


* A Lemma is a proposition preliminarily established, in order to derive another from it.

$\angle DGB = \angle DCB$ (III. Prop. 21). But it has been proved that $\angle DAF$ is equal to $\angle DGF$ or DGB ; and therefore $\angle DAF = \angle DCB$ (Ax. 1). But the angle B is common to the two triangles BAE , DCB , and therefore the remaining angles BDC , BEA are also equal (I. Prop. 32). But $\angle BDC$ is a right angle (Hyp.), consequently $\angle BEA$ is also a right angle, and AE is perpendicular to BC .

Hence it follows that the three perpendiculars drawn from the angles of any triangle to the opposite sides, must meet in the same point. For if it be supposed that the perpendicular from the angle A does not pass through the point F in which the perpendiculars from the angles B and C meet or intersect, then join AF and produce it to E , and AE must be perpendicular to BC according to the preceding Lemma. Therefore there are two perpendiculars drawn from the point A to the line BC , which is absurd. There can be drawn therefore no perpendicular from A to BC but that which passes through F .

The foregoing Lemma may be also demonstrated without reference to the Third Book. For, bisect AF in P ; join DP , GP , and produce the latter towards L . Then since ADF is a right-angled triangle, $DP = PF = PA$ (S. Prop. 16); and for a like reason $GP = PF = PA$. Then since FPG is an isosceles triangle, the external angle LPF is double of $\angle PGF$ (I. Prop. 32); and, in like manner, $\angle LPD$, which is a part of $\angle LPF$ is double of $\angle PGD$, which is a part of $\angle PGF$; and therefore $\angle DPF$, the remainder of $\angle LPF$, is double of $\angle DGF$, which is the remainder of $\angle PGF$. But $\angle DPF$ is also double of $\angle DAP$ (I. Prop. 32), and therefore $\angle DAP = \angle DGF$. Again, bisect BC in H ; join DH , GH , and produce GH to K , then DH and GH are each equal to half of the base BC (S. Prop. 16), and are therefore equal to each other. Consequently $\angle DHK$ is double of $\angle DGH$; but $\angle BHK$, which is part of $\angle DHK$, is also double of $\angle BGH$, which is part of $\angle DGH$; there-

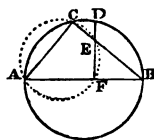


fore $\angle DHB$, which is the remaining part of $\angle DHK$, is double of $\angle DGB$, which is the remainder of $\angle DGH$. But $\angle DHB$ is also double of $\angle DCH$, and therefore $\angle DGB = \angle DCB$. But it has been already proved that $\angle DAP$ or $\angle DAE = \angle DGF$ or $\angle DGB$, and since $\angle DGB = \angle DCB$, it follows that $\angle DAE$ or $\angle BAE = \angle DCB$. The triangles BAE , DCB having those angles equal, and the angle at B common, $\angle AEB$ also must be equal to $\angle CDB$ (I. Prop. 32), and is therefore a right angle.

PROP. LXXXIII.

If a line (DF) be drawn at right angles to the diameter (AB) of a circle (ADB), and cutting a chord (BC) drawn from one extremity of that diameter, the rectangle ($CB \cdot BE$) under the chord and its segment conterminous with the diameter, will be equal to the rectangle ($AB \cdot BF$) under the diameter and its segment adjoining the chord.

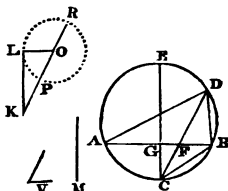
Since $\angle ACB$ is a right angle (III. Prop. 31), and also $\angle AFD$ (Hyp.), the quadrilateral figure $ACEF$ is capable of having a circle circumscribed about it (III. Prop. 22); of which circle BA and BC are both secants, and consequently the rectangles $BA \cdot BF$, $BC \cdot BE$, being both equal to the square of the tangent drawn to that circle from the point B (III. Prop. 36), are equal to each other (Ax. 1).



PROP. LXXXIV.

Given the base (AB) and vertical angle (V) of a triangle, and the line (M) bisecting that angle, to construct the triangle.

On the given line AB describe a segment of a circle containing an angle equal to the given angle V (III. Prop. 33); and complete the circle. Draw the diameter EC perpendicular to AB, and join CB. Then assume a line, KL, equal to CB; from one of its extremities draw the perpendicular LO, and make it equal to half of the given line M. With O as centre and OL as radius describe the circle PLR, and through its centre draw from K the secant KR. Then in the circle AEB place from the point C the chord CD equal to KR; join AD, BD, and ADB will be the required triangle.

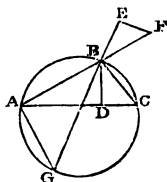


For the rectangle $EC \cdot GC = DC \cdot FC$ (8. Prop. 83); but $EC \cdot GC = CB^2$ (1. Prop. 47), and therefore also $DC \cdot FC = CB^2$. But since $CB = KL$ (Const.), their squares are equal, and $KL^2 = KR \cdot KP$ (III. Prop. 36), therefore also $CB^2 = KR \cdot KP$; and since $DC \cdot FC = CB^2$, it follows, that $DC \cdot FC = KR \cdot KP$. But $CD = KR$ (Const.), and therefore also $CF = KP$, and consequently, $FD = PR$. But PR , being the diameter, is double of the radius DO , which is equal to half of M (Const.); therefore $PR = M$, and consequently, $FD = M$. The triangle ADB therefore, is on the given base AB , has an angle equal to the given one, and the line DF , which bisects that angle (III. Prop. 27), is equal to the given line M .

PROP. LXXXV.

The rectangle contained by any two sides (AB, BC) of a triangle (ABC) is equal to the rectangle under the diameter (BG) of the circumscribing circle and the perpendicular (BD) drawn to the third side (AC) from the opposite angle.

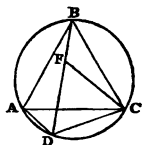
Produce the diameter BG till its produced part BE is equal to BD; and produce the side AB till its produced part BF is equal to the other side BC, and join EF and AG. Then, since $\angle AGB = \angle ACB$, and $\angle s$ BAG, BDC are both right angles, $\angle s$ ABG, DBC must be also equal. But $\angle ABG = \angle EBF$ (I. Prop. 15); therefore $\angle EBF = \angle DBC$; and since the sides containing these angles are equal, viz. BE to BD, BF to BC (Const.), the triangles EBF, DBC are equal (I. Prop. 4); and $\angle BEF$ being equal to $\angle BDC$, is a right angle. Consequently, since $\angle s$ GAF, GEF are both right angles, they may be both contained in the semicircle described on GF, in which semicircle, the lines AF, GE being intersecting chords, make equal rectangles under their segments (III. Prop. 35), that is to say, $AB \cdot BF = GB \cdot BE$. But $BF = BC$ (Const.), and in like manner, $BE = BD$, and consequently, $AB \cdot BC = GB \cdot BD$, or the rectangle contained by two sides of the triangle is equal to the rectangle contained by the diameter of the circumscribing circle, and the perpendicular let fall on the third side from the opposite angle.



PROP. LXXXVI.

If an equilateral triangle (ABC) be inscribed in a circle, and chords (AD , BD , CD) be drawn from its angles to any one point (D) in the circumference, that chord (BD) which falls within the triangle will be equal to the other two (AD , CD).

On DB take DF equal to DC , and join FC . Then, $\angle FDC$ being equal to $\angle BAC$ (iii. Prop. 21), is equal to two-thirds of a right angle; therefore, $\angle$ s DFC , DCF being equal (i. Prop. 5), are also each equal to two-thirds of a right angle (i. Prop. 32), and FCD is an equilateral triangle. Therefore $\angle FCD = \angle BCA$, and taking from both the common part, $\angle FCA$, there remains $\angle ACD = \angle BCF$; and these equal angles are contained by equal sides, viz. $BC = AC$, and $CF = CD$; therefore the triangles BCF , ACD are equal (i. Prop. 4), and the base $BF = AD$. Therefore $BF + FD = BD = AD + DC$, or the chord passing through the triangle is equal to the other two.

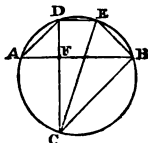


PROP. LXXXVII.

If two chords (AB , CD) meet at right angles within or without a circle, the squares of their segments (AF , FB , CF , FD) intercepted between their point of meeting (F) and the circumference, are together equal to the square of the diameter (CE).

From either extremity of one of the chords, as CD , draw the diameter CE , and join AD , DE , EB and CB . Then, $AD^2 = DF^2 + FA^2$ (i. Prop. 47); and, in like manner,

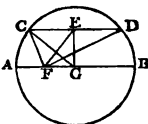
$CB^2 = CF^2 + FB^2$; therefore AD^2 and CB^2 together equal the squares of the segments of the chords. But since $\angle CDE$ is a right angle (iii. Prop. 31), DE is parallel to AB (i. Prop. 27), and therefore $EB = AD$ (s. Prop. 70), and consequently, EB^2 and CB^2 are together equal to the squares of the segments of the chords; but they are also equal to CE^2 (i. Prop. 47); and therefore the square of the diameter is equal to the sum of the squares of the segments of the chords.



PROP. LXXXVIII.

If from any point (F) in the diameter (AB) of a circle, or in the diameter produced, straight lines (FC, FD) be drawn to the extremities of a chord (CD) parallel to the diameter, the sum of their squares will be equal to the sum of the squares of the segments of the diameter.

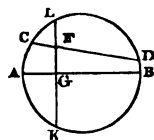
Bisect CD in E , and join FE , GE , and GC . Then GE will be perpendicular to CD (iii. Prop. 3); and since CD is parallel to AB (Hyp.), $\angle EGF$ will be a right angle (i. Prop. 29). But $FC^2 + FD^2 = 2 FE^2 + 2 CE^2$ (s. Prop. 64); and since $FE^2 = EG^2 + FG^2$ (i. Prop. 47), $2 FE^2 + 2 CE^2 = 2 EG^2 + 2 FG^2 + 2 CE^2$; but $CE^2 + EG^2 = GC^2$ (i. Prop. 47); therefore $2 FE^2 + 2 CE^2 = 2 GC^2 + 2 FG^2$; and $GC = GA$, therefore $2 FE^2 + 2 CE^2 = FC^2 + FD^2 = 2 GA^2 + 2 FG^2$. But the squares of any two lines are equal to twice the squares of half their sum and half their difference (ii. Prop. 9). Therefore $AF^2 + FB^2 = 2 GA^2 + 2 FG^2 = FC^2 + FD^2$, or the squares of the segments of the diameter are equal to the squares of the lines drawn from the point of section to the extremities of the parallel chord.



PROP. LXXXIX.

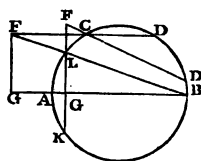
If from any point (G) in the diameter (AB) of a circle, or diameter produced, a perpendicular (GF) be drawn meeting any chord (CD), or chord produced, the rectangle contained by the segments of the diameter, is equal to the sum or difference of the square of the perpendicular and the rectangle contained by the segments of the chord, according as the perpendicular meets the chord within or without the circle.

First, let the points G and F, in which the perpendicular meets the diameter and the chord, both fall within the circle. Then the rectangle $AG \cdot GB = GK^2$ (iii. Props. 3, 35); but $GK^2 = LF \cdot FK + GF^2$ (ii. Prop. 5); and also $LF \cdot FK = CF \cdot FD$ (iii. Prop. 35). Therefore the rectangle $AG \cdot GB = CF \cdot FD + GF^2$.



If the point F falls without the circle, while G falls within; then $FG^2 = KF \cdot FL + LG^2$ (ii. Prop. 6); but $KF \cdot FL = DF \cdot FC$ (iii. Prop. 36), and $LG^2 = AG \cdot GB$; and therefore $FG^2 = DF \cdot FC + AG \cdot GB$, or $DF \cdot FC = FG^2 - AG \cdot GB$.

Next, suppose the points F and G to fall both without the circle. Then join F and the extremity of the diameter B; and also join AL. In this case, GB^2 is the difference between FB^2 and FG^2 , or $GB^2 = FB^2 - FG^2$ (i. Prop. 47); but $GB^2 = GB \cdot GA + GB \cdot AB$ (ii. Prop. 3); and, in like manner, $FB^2 = FB \cdot FL + FB \cdot LB$; also, $GB \cdot AB = FB \cdot LB$ (s. Prop. 83); and if these equals be taken from GB^2 and FB^2 , there will remain $GB \cdot GA = FB \cdot FL - FG^2$. But $FB \cdot FL = FD \cdot FC$; and therefore $GB \cdot GA = FD \cdot FC - FG^2$.



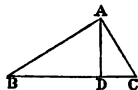
SIXTH BOOK.

PROP. XC.

The hypotenuse and ratio of the sides of a right-angled triangle (BAC) being given, to construct the triangle.

Divide the hypotenuse BC , so that the ratio of its segments BD , DC may be the duplicate of the given ratio of the sides; and from the point D draw the perpendicular DA , and make it a mean proportional between the segments of the hypotenuse. Draw AB , AC ; and BAC will be the required right-angled triangle.

For since $BD : DA :: DA : DC$ (Const.), and $\angle ADB = \angle ADC$ (Const.), the triangles BDA , ADC , having one angle in the one equal to one angle in the other, and these equal angles contained by proportional sides, are equiangular (vi. Prop. 6), and their sides about equal angles are proportional (vi. Prop. 4), the equal angles being opposite to homologous sides. Therefore $\angle BAD = \angle ACD$, and $\angle CAD = \angle ABD$. Consequently, $\angle s$ BAD , CAD together, that is to say, the vertical angle BAC , is equal to the sum of the angles ABC , ACB at the base, and therefore $\angle BAC$ must be a right angle (i. Prop. 32). But it is manifest, that the triangles BAC and ADB are equiangular, and therefore $BA : AC :: BD : DA$ (vi. Prop. 4). And since $BD : DA :: DA : DC$ (Const.), and $BD : DC$ is the duplicate of the given ratio of the sides, $BD : DA$ is equal to the given ratio. But $BA : AC : BD : DA$, and therefore the sides of the right-angled triangle BAC are in the required ratio.



PROP. XCI.

The altitude (AD) and ratio of the sides (BA : AC) being given, to construct a right-angled triangle.

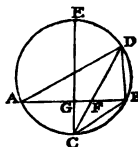
[Preceding Diagram.]

Through one extremity D of the given altitude draw an indefinite straight line at right angles with it. Let the given ratio be $x : y$; then, in the indefinite straight line take the part DB, so that $AD : DB :: y : x$; and also take DC, so that $AD : DC :: x : y$. Then it is evident that DB or $BD : AD :: x : y :: AD : DC$; and if AB and AC be drawn, it may be demonstrated, in the same terms as in the preceding Proposition, that $\angle BAC$ is a right angle, and that the sides containing it are in the given ratio.

PROP. XCII.

To construct a triangle, of which the base, the ratio of the sides, and the vertical angle are given.

On the given base AB describe a segment of a circle, containing an angle equal to the given vertical angle (III. Prop. 33). Complete the circle, and draw the diameter EC making right angles with AB. Divide AB so that its segments AF and FB may be in the given ratio of the sides of the triangle. From C, through F, draw CD to the circumference, and if AD, BD be drawn, ABD will be the required triangle.



For since the diameter EC makes right angles with AB, it must also bisect it (III. Prop. 3), and thence it may be easily shown that the chords of the arches AC, BC are

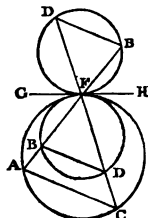
equal (I. Prop. 4), and consequently, that the arches themselves are equal (III. Prop. 28). Therefore $\angle$ s ADC, BDC, standing on them at the circumference, are equal (III. Prop. 27), and CD bisects the angle ADB. Therefore, the sides are proportional to the segments of the base, or $AD : DB :: AF : FB$ (vi. Prop. 3); but $AF : FB$ is the given ratio (Const.); and consequently, the triangle ADB, constructed on the given base, and with its vertical angle equal to the given one, has also its sides containing that angle in the given ratio.

PROP. XCIII.

If two circles (FAC, FDB) touch in any point, the straight lines (AB, CD), drawn from the point of contact and meeting both circumferences, will be cut proportionally by them.

For if the straight lines AC and BD be drawn, they must be parallel to each other (8. Prop. 72), and therefore the triangles AFC, BFD are equiangular, and have their sides proportional (vi. Prop. 4), or $AF : FB :: CF : FD$. But these proportionals are the segments of the lines drawn through the point of contact of the circles.

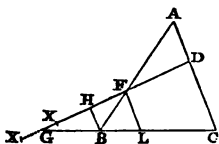
If a diameter of one of the circles be drawn through the point of contact, it must pass through the centre of the other circle also (III. Props. 11, 12), and then it will be manifest, that all lines drawn through the point of contact to the circumferences of the circles, will be proportional to the diameters.



PROP. XCIV.

If a line (DG) cutting two sides of a triangle (ABC) meet the base produced, the whole produced base will be to the produced part in a ratio compounded of the ratios of the segments of the sides.

Draw BH parallel to AC; then since the lines CD, BH are parallel, $CG : BG :: CD : BH$ (vi. Prop. 2). But the ratio of CD to BH may be conceived to be compounded of the ratios of CD to DA, and of DA to BH, and $DA : BH :: AF : FB$ (vi. Prop. 2); therefore the ratio CD : BH is compounded of the ratios CD : DA and AF : FB. But it has been shown, that $CG : BG :: CD : BH$, and therefore the ratio CG : BG is compounded of the ratios CD : DA and AF : FB, or the whole produced base is to the produced part in a ratio compounded of the ratios of the segments of the sides.



COROLLARY 1.—The whole cutting line is to its external segment, or $DG : FG$, in a ratio compounded of the ratios $CD : CA$ and $AB : FB$.

For draw FL parallel to CA, then $DG : FG :: DC : FL$ (vi. Prop. 2); but $DC : FL$ is a ratio compounded of the ratios $CD : CA$ and $CA : FL$; and $CA : FL :: AB : FB$ (vi. Prop. 2); therefore the ratio $DC : FL$ is compounded of the ratios $CD : CA$ and $AB : FB$; but $DG : FG :: DC : FL$, and therefore the ratio $DG : FG$ is compounded of the ratios $CD : CA$ and $AB : FB$.

COR. 2.—If a line DF, cutting the sides of a triangle CAB, be produced till the whole produced line be to the produced part in a ratio compounded of the ratios of the sides to their segments towards the base, that is to say,

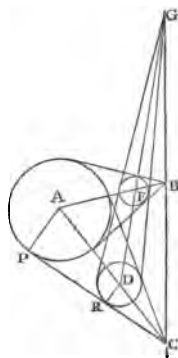
of the ratios $CD : CA$ and $AB : FB$; then the base CB being produced shall meet the extremity of that cutting line.

For if it be supposed that the produced base does not meet the extremity X of the cutting line drawn as described above, but either cuts it or meets it still further produced, in G ; then the ratio $DG : FG$ is compounded of the ratios $CD : CA$ and $AB : FB$ (Cor. 1); but the ratio $DX : FX$ is compounded of the same ratios (Hyp.); therefore $DG : FG :: DX : FX$, and by conversion (v. Def. 18), and inversion (vi. Prop. 6, A) $DG - FG : FG :: DX - FX : FX$ (vi. Prop. 19, D), that is to say, $DF : FG :: DF : FX$, and therefore $FG = FX$ (vi. Prop. 9), which is absurd. Therefore the produced base neither cuts the cutting line definitely produced as above described, nor does it meet it still further produced; it must therefore meet it at its extremity.

PROP. XCV.

If three unequal circles (A, D, F) be joined by three pairs of tangents, the three points of meeting (C, B, G) of these tangents will be in the same straight line.

From C , the point of meeting of the tangents drawn to the circles A and D , draw CD to the centre of the latter; and it is evident that the angles made by CD with the tangents will be equal; for they are subtended, in equal triangles (i. Prop. 8), by equal sides, viz. the rays of the circle D . In like manner, the line CA , drawn to the centre of the circle A , must bisect the angle made by the tangents; and therefore CD produced must pass through A , since there cannot be two lines bisecting the same angle. In the same manner it

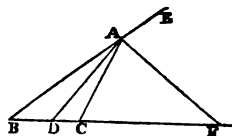


may be proved, that the lines joining the centres of the circles A and F, and of the circles D and F respectively, pass through the points in which the tangents joining those circles meet. But if from the centres A and D, be drawn rays to the points of contact P and R, these rays, AP and DR, being both perpendicular to the tangent PC (iii. Prop. 18), are parallel to each other (i. Prop. 28), and therefore $CD : CA :: DR : AP$ (vi. Prop. 2). In like manner it may be shown that $AB : FB :: AP : \text{ray of circle F}$; and also, that $DG : FG :: \text{ray of D} : \text{ray of F}$. But the ray of D is to the ray of F in a ratio compounded of the ratios of the ray of D to the ray of A, and of the ray of A to the ray of F; and therefore the ratio $DG : GF$ is compounded of the ratios $CD : CA$ and $AB : FB$; and consequently, the line joining the points C and B must, when produced, pass through the point G (s. Prop. 94, Cor. 2).

PROP. XCVI.

Lines (AD, AF) bisecting an angle of a triangle and the external angle adjacent to that angle, cut the base (BC) harmonically, that is to say, they cut it internally and externally in the same ratio.

The line AD bisecting the internal angle divides the base into segments proportional to the sides, so that $BD : DC :: BA : AC$ (vi. Prop. 3); and the line AF bisecting the external angle, likewise cuts the produced base into segments proportional to the sides, so that $BF : CF :: BA : AC$. Therefore $BD : DC :: BF : CF$, or the base is cut internally and externally in the same ratio.



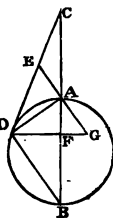
Hence by alternation (v. Prop. 16), and inversion (v. Prop. 6, A), $BF : BD :: CF : DC$. From this it is obvious, that since the lines AD and AF contain a right angle, the Propo-

sition may be thus expressed, viz. If, in a right-angled triangle, lines be drawn from the vertex of the right angle, making equal angles with one of the sides, these lines, if they severally meet the hypotenuse and hypotenuse produced, divide it internally and externally in the same ratio.

PROP. XCVII.

If a tangent (CD) be drawn to a circle (ABD) from any point (c)-in its diameter produced, and from the point of contact a perpendicular (DF) be drawn to the diameter, the tangent and perpendicular divide the diameter internally and externally in the same ratio.

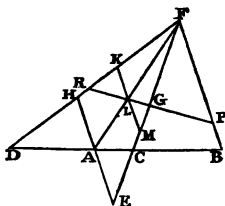
Join DA and DB, and through A draw EG parallel to DB, and meeting DF produced to G. Then, $\angle CDA$ made by the tangent and secant is equal to $\angle DBA$ in the alternate segment (iii. Prop. 32). But the right-angled triangles ADB, AFD, having $\angle DAF$ in common, are equiangular, and $\angle ADF = \angle DBA$; therefore $\angle CDA = \angle ADF$ or $\angle ADG$. But since EG is parallel to DB, it makes right angles with DA; and therefore the triangle EDG, of which DA bisects the vertical angle, and is perpendicular to the base, is isosceles (s. Prop. 12), and consequently its base EG is bisected by DA (s. Prop. 1), and $EA = GA$. But $DB : EA :: BC : CA$ (vi. Prop. 2); and again, $BD : GA :: BF : FA$; therefore $BC : CA :: BF : FA$ (v. Prop. 11), or the diameter produced is cut internally and externally in the same ratio.



PROP. XCVIII.

If a straight line (AB) be divided harmonically, and from its extremities (A, B) and points of section (D, C) straight lines be drawn to one point (F), any straight line (PR) drawn across those straight lines will also be divided harmonically.

Through the point A draw the straight line HE parallel to FB, and meeting FD, and also FC produced to E. Then $FB : HA :: BD : DA$ (vi. Prop. 2); and also $FB : AE :: BC : CA$; but $BD : DA :: BC : CA$ (Hyp.), and therefore $FB : HA :: FB : AE$, and consequently, $HA = AE$ (v. Prop. 9). Now, through the point L, in which the line RP cuts FA, draw KM parallel to HE, and it is evident, that KM will be divided in the same proportion as HE; that is to say, it will be bisected, and $KL = LM$. But $FP : KL :: PR : RL$; and also $FP : LM :: PG : GL$; and since $KL = LM$, $FP : KL :: FP : LM$ (v. Prop. 7); therefore $PR : RL :: PG : GL$, and the line PR is divided harmonically.

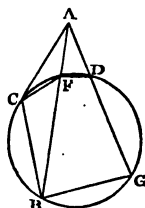


PROP. XCIX.

If from the same point (A) two lines be drawn, the one (AC) touching a circle, the other (AB) cutting it, the tangent will be a mean proportional between the whole secant and its external segment.

For if CF and CB be drawn from the point of contact to the points of section, the triangles ACF, BAC, having

$\angle CAB$ common, and also $\angle ACF = \angle ABC$ (iii. Prop. 32), must have their remaining angles equal, and, being equiangular, have their sides about equal angles proportional (vi. Prop. 4); and therefore $BA : AC :: AC : AF$. Consequently, the tangent AC is a mean proportional between the whole secant AB and its external segment AF .



The same conclusion may be derived from a different principle. Since AC touches the circle, and AB cuts it, $AC^2 = AB \cdot BF$ (iii. Prop. 36), and therefore $BA : AC :: AC : AF$ (vi. Prop. 17).

PROP. C.

If from the same point (A) several straight lines be drawn cutting a circle, these whole secants (AB, AG) will be to one another inversely as their external segments (AF, AD).

[Preceding Diagram.]

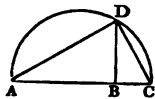
Join FD and BG ; and since $FDBG$ is a quadrilateral figure inscribed in a circle, $\angle s$ BFD, BGD are equal to two right angles (iii. Prop. 22). But also $\angle s$ BFD, AFD are equal to two right angles (i. Prop. 13); and therefore $\angle AFD = \angle BGD$. The triangles ADF, ABG , therefore, having the angle BAG common, and also $\angle s$ AFD, BGD equal, are equiangular (i. Prop. 32), and have their sides about equal angles proportional (vi. Prop. 4). Consequently, $BA : AG :: AD : AF$.

Or the Proposition may be thus proved. Since AC^2 is equal to $BA \cdot AF$, and also to $GA \cdot AD$ (iii. Prop. 36), it follows, that $BA \cdot AF = GA \cdot AD$, and therefore $BA : GA :: AD : AF$ (vi. Prop. 16).

PROP. CI.

Any chord (AD) in a circle is a mean proportional between the diameter (AC) and the segment (AB) of the diameter drawn from one of its extremities, intercepted between that extremity and the perpendicular (DB) let fall on it from the other extremity of the chord.

For join DC; and then ADC is a right-angled triangle (iii. Prop. 31), and consequently equiangular with the triangle ABD, which has one angle, viz. $\angle BAD$, in common with it. Therefore $AC : AD :: AD : AB$ (vi. Prop. 4), or the chord is a mean proportional between the diameter and its segment conterminous with the chord.

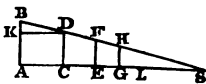


It is obvious, from the similarity of the triangles ADC, ABD, DBC (vi. Prop. 8), that if, in a right-angled triangle, a perpendicular be let fall from the vertex of the right angle on the hypotenuse, the segments of the hypotenuse are in the duplicate ratio of the sides of the triangle.

PROP. CII.

Given the first two terms of a decreasing series of lines in geometrical progression, to find the sum.

Assume an indefinite straight line, and on it take the parts AC and CE, respectively equal to the given terms of the series. From A draw AB perpendicular to AS, and make it equal to AC; and in like manner, draw from C the perpendicular CD equal to CE. Join BD, and produce it till it meets the assumed indefinite



straight line in S. AS is the sum of the series. For draw EF parallel to CD; and since CD is parallel to AB, $AB:AS::CD:CS$ (vi. Prop. 2); and by inversion (v. Prop. 6, A) and conversion (v. Prop. 19, D) $AB:AS-AB::CD:CS-CD$; or since $AC=AB$, and $CE=CD$ (Const.), $AB:CS::CD:ES$; and by alternation (v. Prop. 16), $AB:CD::CS:ES$. But $CS:ES::CD:EF$ (vi. Prop. 2), and therefore $AB:CD::CD:EF$, or EF is a third proportional to the given terms. And if EG be taken on the line ES equal to EF, and GH be drawn parallel to EF, GH may be in like manner shown to be the fourth of the series of proportionals. If GL be taken equal to GH, the perpendicular drawn from it, and meeting the line BS, will be the fifth term of the series. Now, since $GH:GS::AB:AS$, and it is evident from the construction that AB is less than AS, so also GH must be less than GS; and the perpendiculars AB, CD, EF, GH, or their equals AC, CE, EG, GL being taken from the line AS to any assignable extent, must still leave a remainder. Their sum, therefore, can never exceed AS; but as the remainders continually decrease, and the series may be continued infinitely, the last remainder must be less than any assignable magnitude, and therefore AS is the sum of the series.

PROP. CIII.

The sum of a decreasing series in continued proportion or geometrical progression has a limit, though the number of the terms has not.

[Preceding Diagram.]

This important principle, though the demonstration of it is contained in that of the preceding Proposition, yet merits a separate enunciation, in order to impress it on the mind of the learner. It is evident, that the sum of the series AB, CD, EF, &c., since it can never exceed

AS, has a limit; and also, that since the last found term of the series, no matter how far the series may have been continued, has the same ratio to the last remainder, as AB has to AS, another term may still be added to the series, and consequently that the series is infinite.

If from D be drawn DK parallel to CA, it is evident that it will be equal to CA (I. Prop. 34), and therefore equal also to AB (Const.). And since $KA = DC$ (I. Prop. 34), BK is equal to the difference between AB and CD. But $BK : KD :: BA : AS$ (VI. Prop. 2), or, since $KD = BA$, $BK : BA :: BA : AS$; that is to say, the difference between the first two terms is to the first term, as the first term is to the sum of the series. Hence we can always find the sum of a decreasing series from the first two terms. If, for instance, 9 and 6 be the first two terms of a series of numbers in continued proportion, then we have the analogy $(9-6)$ or $3 : 9 :: 9 : 27$. Therefore 27 is the sum of the series; and it will be found on trial, that the series of proportional numbers, 9, 6, 4, $2\frac{2}{3}$, $1\frac{1}{3}$, &c., though continually approximating to, yet never actually attains the sum of 27.

PROP. CIV.

Of three terms in continued proportion, the mean and the sum of the extremes being given, to find the extremes.

If the proportionals be lines; take a line equal to the given sum of the extremes, and describe on it a semicircle. From the extremity of the same line draw a perpendicular equal to the given mean, and from the extremity of that perpendicular draw a line parallel to the first assumed line. From one of the points in which that parallel line cuts the semicircle, draw a line parallel to the perpendicular, and meeting the first assumed line. The

line last drawn divides the assumed line into parts equal to the extremes. For the dividing line is equal to the perpendicular (i. Prop. 34), which is equal to the given mean (Const.); therefore the dividing line is equal to the given mean. But being in a semicircle, it is the altitude of a right-angled triangle (iii. Prop. 31), and is therefore a mean proportional between the segments of the hypotenuse (vi. Prop. 13); and since the hypotenuse, in this case, is equal to the given sum of the extremes (Const.), it is evident that the segments into which the assumed line is divided, are equal to the extremes.

If the proportionals be numbers, the solution of this Problem may be conveniently derived from the principle that the square of the mean is equal to the rectangle contained by (or the product of) the extremes (vi. Prop. 17). From the square of the given sum of the extremes, take four times the square of the mean, and the square root of the remainder, added to the given sum, will be double of the greater extreme: the remainder of the sum will be the less.

For the square of the whole line or sum is equal to four times the rectangle under its segments (that is, the extremes) together with the square of the difference of the segments (ii. Prop. 8). Four times that rectangle therefore, or four times its equal, the square of the mean, being deducted from the square of the sum, leaves the square of the difference. The segments or extremes then are easily found from their sum and difference.*

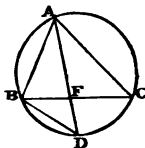
To illustrate this method of proceeding, let us suppose that of three proportionals the mean is 6, and the sum of the extremes 13. From the square of 13, or $13 \times 13 = 169$, take four times the square of 6, or $4 \times 6 \times 6 = 144$, and $169 - 144 = 25$; and the square root of 25 is 5. But the half of $13 + 5 = 18$ is 9, and the half of $13 - 5 = 8$ is 4; therefore the three proportionals are 9, 6, and 4.

* See Appendix to Cooley's Euclid, p. 165.

PROP. CV.

In any triangle (ABC), the rectangle contained by any two of its sides ($AB \cdot AC$) is equal to the square of the line (AF) bisecting the angle contained by them, together with the rectangle contained by the segments of the base.

Describe a circle $ABDC$ round the given triangle (iv. Prop. 5): produce AF to the circumference in D , and join BD . Then, since $\angle s$ BAD , CAF are equal (Hyp.), and $\angle s$ ACF , ADB are also equal (iii. Prop. 21), the triangles ABD , AFC have their remaining angles equal (i. Prop. 32), and are equiangular. They have their sides, therefore, about equal angles proportional (vi. Prop. 4), and $BA : AD :: AF : AC$; consequently, since the rectangle under the extremes is equal to the rectangle under the means (vi. Prop. 16), $BA \cdot AC = AD \cdot AF$. But $AD \cdot AF = AF^2 + AF \cdot FD$ (ii. Prop. 3), and $AF \cdot FD = BF \cdot FC$ (iii. Prop. 35); consequently, $AD \cdot AF = AF^2 + BF \cdot FC$, and since $BA \cdot AC = AD \cdot AF$, it follows, that $BA \cdot AC = AF^2 + BF \cdot FC$, or the rectangle contained by the sides of the triangle, is equal to the square of the line bisecting the contained angle, together with the rectangle under the segments of the base.

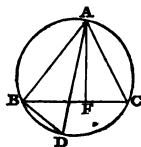


PROP. CVI.

The rectangle contained by any two sides (AB , AC) of a triangle (ABC) is equal to the rectangle contained by its altitude (AF), or perpendicular drawn to the third side from the opposite angle, and the diameter (AD) of the circumscribing circle.

Describe a circle $ABDC$ round the given triangle, draw from A the perpendicular AF , the diameter AD , and join

BD. Then since the triangles ABD, AFC have each a right angle, and have $\angle$ s ADB, ACF also equal (III. Prop. 21), they are equiangular, and similar (VI. Def. 1), and therefore $AB:AD::AF:AC$, and consequently $AB \cdot AC = AD \cdot AF$ (VI. Prop. 16), or the rectangle under the sides of the triangle is equal to the rectangle under its altitude and the diameter of the circumscribing circle.



PROP. CVII.

Of all the triangles which may be inscribed in a given circle, the equilateral is the greatest.

For the area of a triangle is equal to the rectangle under its altitude and half of its base (I. Prop. 41), and therefore when the base is given, the triangle varies as the altitude; consequently, the greatest triangle contained in a given segment of a circle, is the triangle of which the altitude passes through the centre of the circle, and which is therefore isosceles. But if this isosceles triangle be not also equilateral, then one of the equal sides being assumed as the base, a greater triangle may be inscribed in the segment of the circle, standing on that base. And in general, while any two sides are unequal, a greater triangle may be inscribed in the segment standing on the third side. The inscribed equilateral triangle alone cannot be exceeded, and is therefore the greatest triangle which can be inscribed in a circle.

Let M and N represent the sides of a triangle, B the base, A the altitude, and D the diameter of the circumscribing circle, then $M \cdot N = A \cdot D$ (s. Prop. 106); multiply both by B, and $M \cdot N \cdot B = A \cdot D \cdot B$; and again divide both by $2D$, and $\frac{M \cdot N \cdot B}{2D} = \frac{A \cdot B}{2}$, or the continued product of the three sides of a triangle, divided by twice the diameter of the circumscribing circle, is equal to the area of the triangle. The area, therefore, of a triangle inscribed in a given

circle, varies as the continued product of the sides, which can be easily shown to be the greatest in the case of the equilateral triangle.

PROP. CVIII.

If chords be drawn in any segment of a circle from its extremities to a point in the circumference, the rectangle contained by them will be greatest when they are equal.

For if a line be drawn joining the extremities of the segment, it will be the base of the triangle of which the chords are the sides; and the rectangle under the chords will be equal to the rectangle under the diameter of the circle and the altitude of that triangle (s. Prop. 106): but it is evident that the altitude will be greatest when the sides of the triangle, that is to say, when the chords, are equal.

PROP. CIX.

Of quadrilateral figures inscribed in a given circle, the square is the greatest.

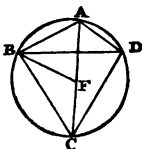
For draw the diagonals of the figure; and then if any two sides be unequal, there may be constructed on the diagonal which joins their extremities, an isosceles triangle in the circle which will be greater than the triangle of which they are the sides. Consequently, if a quadrilateral figure inscribed in a circle have any of its sides unequal, a larger quadrilateral figure may be inscribed in the same circle. But the square cannot be thus exceeded, and is therefore the largest quadrilateral figure which can be inscribed in a circle.

It is obvious that the same mode of reasoning may be applied to figures of any number of sides. The largest pentagon, therefore, or hexagon inscribable in a circle is that which is regular, or has all its sides equal.

PROP. CX.

In a quadrilateral figure (ABCD) inscribed in a circle, the rectangle under the diagonals (AC, BD) is equal to the sum of the rectangles under the opposite sides.

Draw BF making with the side BC, $\angle$ FBC equal to $\angle$ ABD. Then the triangles DAB, BFC, having the $\angle$ s ABD, FBC equal (Const.), and also $\angle$ s ADB, FCB (iii. Prop. 21), are similar, and therefore $AD : DB :: FC : CB$ (vi. Prop. 4), and consequently $AD \cdot CB = DB \cdot FC$. And again, if to each of the equal angles, ABD, FBC, be added $\angle$ DBF, then the sums, $\angle$ s ABF, DBC, will be equal (Ax. 2), and also $\angle$ s FAB, CDB are equal (iii. Prop. 21). Therefore the triangles FAB, CDB are similar, and $AB : AF :: DB : CD$, and therefore $AB \cdot CD = AF \cdot DB$. But we have seen also that $AD \cdot CB = DB \cdot FC$; consequently, if these results be added together, $AB \cdot CD + AD \cdot CB = DB \cdot FC + DB \cdot AF$. But $DB \cdot FC + DB \cdot AF = DB \cdot (AF + FC)$, or $DB \cdot AC$; and therefore $AB \cdot CD + AD \cdot CB = DB \cdot AC$, or the rectangles under the opposite sides, are together equal to the rectangle under the diagonals.



PROP. CXI.

To construct on a given line a right-angled triangle, having its sides in continued proportion or geometrical progression.

On the given line describe a semicircle, and divide the line in extreme and mean ratio (vi. Prop. 30). From the point of division draw a line perpendicular to the given line,

and meeting the circumference of the semicircle. If lines be drawn from the point where the perpendicular meets the circumference to the extremities of the given line, the right-angled triangle thus constructed will have its sides in continued proportion.

For the whole given line or hypotenuse is to its greater segment, as the greater segment to the less (Const.). The greater segment is therefore a mean proportional between the whole line and the less segment. And again, the whole line is to the less side of the triangle as the less side to the less segment (vi. Prop. 8). The less side, therefore, as well as the greater segment, is a mean proportional between the whole line and the less segment. Consequently, the greater segment of the hypotenuse is equal to the less side of the triangle (v. Prop. 9). But the whole line is to the greater side as the greater side to the greater segment; and since the greater segment is equal to the less side of the triangle, it follows, that the hypotenuse is to the greater side as the greater side is to the less; and therefore the three sides of the triangle are in continued proportion.

It is obvious, that all triangles the sides of which are in continued proportion, are similar, since their sides are in a determinate ratio, viz. the subduplicate of the ratio of the segments of a line cut in extreme and mean ratio.

PROP. CXII.

Given the altitude of a right-angled triangle, the sides of which are in geometrical progression, to find the sides.

Since the ratio of the sides of this triangle is known (s. Prop. 111), the segments of the hypotenuse may be found from the altitude. Let $M : N$ represent the ratio of the greater to the less side; G and L the greater and less segments of the hypotenuse, and A the altitude. Then $M : N :: A : L$ (vi. Prop. 8), and $N : M :: A : G$. Consequently,

since A is given, and the ratio $M : N$ is known, G and L or the segments of the hypotenuse may be found; and from the hypotenuse and G , which equals the less side (s. Prop. 111), the greater side may be readily determined.

PROP. CXIII.

Given in numbers, two sides of a triangle and the diameter of the circumscribing circle, to find the third side.

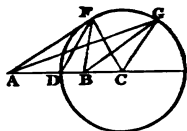
Multiply the two sides together, and divide their product by the diameter. Subtract the square of the quotient from the square of each side, and the square roots of the remainders added together equal the third side.

For the product of the sides equals the product of the diameter and the altitude (s. Prop. 106); and the square of the altitude deducted from the squares of the sides separately, leaves the squares of the segments of the third side (i. Prop. 47); and the segments added together make the third side.

PROP. CXIV.

If two points (A, B) be taken in the diameter of a circle (DFG), so that the rectangle contained by the segments (AC, BC) intercepted between them and the centre of the circle be equal to the square of the radius; and if from these points two straight lines be drawn to any point (F or G) in the circumference of the circle, these lines will have the same ratio to each other ($AF : FB$ or $AG : GB$), as the segments intercepted between the two first-mentioned points and the circumference of the circle ($AD : DB$).

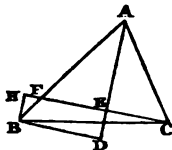
Draw CF , CG , and also DF . Then since $AC \cdot CB = CF^2$ (Hyp.), $AC : CF :: CF : CB$ (vi. Prop. 17), and therefore the triangles AFC , CBF , having the common angle ACF contained by proportional sides, are similar (vi. Prop. 6), and $AF : BF :: AC : CF$ or CD , since $CF = CD$. But since $AC : CF :: CF : CB$, and $CD = CF$, $AC : CD :: CD : CB$, and their differences are in the same ratio, or $AC - CD : CD - CB :: AC : CD$ (v. Prop. 19); that is to say, $AD : DB :: AC : CD$. But it has been shown that $AF : BF :: AC : CD$, and therefore $AF : BF :: AD : DB$. In like manner it may be proved that $AG : GB :: AC : CG$; that $AC : CG :: AD : DB$; and therefore that $AG : GB :: AD : DB$.



PROP. CXV.

If perpendiculars (BD , CE) be drawn from the extremities of the base of a triangle (BAC) to a straight line (AD), which bisects the angle opposite to the base; the area of the triangle is equal to the rectangle contained by either of the perpendiculars and the segment of the bisecting line between the angle and the other perpendicular.

For since AE bisects $\angle FAC$, and is perpendicular to CF (Hyp.), the triangle FAC is isosceles (8. Prop. 12), and the rectangle $AE \cdot EC$ under its altitude and half of its base is equal to its area (1. Prop. 41); or since $FE = EC$ (8. Prop. 1), the rectangle $AE \cdot EF$ is equal to the area of the triangle FAC . And in like manner the rectangle $ED \cdot EF$ is equal to the area of the triangle FBC . Therefore, by addition, $AE \cdot EF + ED \cdot EF = FAC + FBC$, or the triangle ABC . But $AE \cdot EF + ED \cdot EF = (AE + ED) \cdot EF = AD \cdot EF$,



and therefore $AD \cdot EF = ABC$. And since the triangles AEC, ADB are equiangular, $AE : EC :: AD : DB$ (vi. Prop. 4), and therefore $AE \cdot DB = EC \cdot AD$ or $AD \cdot EF$, and therefore also $AE \cdot DB = ABC$.

PROP. CXVI.

The same construction being made as in the preceding Proposition, four times the rectangle contained by the perpendiculars (BD, CE) is equal to the rectangle contained by two straight lines, one of which is the base increased by the difference of the sides, and the other the base diminished by the difference of the sides.

[Preceding Diagram.]

Draw BH parallel to AD , and meeting CF produced to H . Then since FAC is an isosceles triangle, BF is equal to the difference between the sides AB and AC . And since $\angle BHC$ is a right angle (i. Prop. 34), $BF^2 = BH^2 + FH^2$ (i. Prop. 47); and in like manner $BC^2 = BH^2 + CH^2$. Therefore, by subtraction, $BC^2 - BF^2 = (BH^2 + CH^2) - (BH^2 + FH^2)$, or, taking away the common part BH^2 , $BC^2 - BF^2 = CH^2 - FH^2$. But the difference between the squares of two lines is equal to the rectangle under their sum and difference (ii. Props. 5, 6),* and therefore the rectangle under the sum and difference of BC and BF , that is to say, the rectangle under the base increased by the difference of the sides and the base diminished by the difference of the sides, is equal to the rectangle under the sum and difference of CH and FH . But the sum of CH and FH , or $CH + FH = 2 HE = 2 BD$ (i. Prop. 34); and the difference of CH and FH is $FC = 2 EC$. The rectangle, therefore, under the sum and difference of CH and FH ,

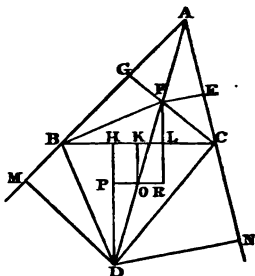
* See Cooley's Euclid, Append. p. 165.

is equal to $2 BD \cdot 2 EC = 4 BD \cdot EC$; and therefore four times the rectangle under the perpendiculars, is equal to the rectangle under the base increased by the difference of the sides, and the base diminished by the difference of the sides.

PROP. CXVII.

The area of any triangle (ABC) is a mean proportional between the rectangle contained by the semiperimeter and its excess above the base, and the rectangle contained by the excesses of the semiperimeter above each of the legs.

Draw AF , BF bisecting $\angle s$ BAC , ABC , and meeting in F ; and also draw BD bisecting the external angle MBC , and meeting AF produced in D . From the point D draw DM perpendicular to AB produced, and meeting it in M ; and DN in like manner perpendicular to AC produced; and also draw DH perpendicular to BC . From F draw FG parallel to DM . Now, since the triangles MAD , NAD have $\angle s$ DAM , DAN equal (Const.), and also $\angle s$ M , N equal (Ax. 10); and since the side AD opposite to equal angles is common to both, the triangles are equal (i. Prop. 26); $DM = DN$ and $AM = AN$.



In like manner, the triangles DBM , DBH are equal (i. Prop. 26), $BM = BH$ and $DM = DH$. And since DH , being equal to DM , must be equal also to DN , and DC is the common hypotenuse of the triangles DCH , DCN , the remaining sides CH , CN of those triangles are equal. But since $BM = BH$ and $CN = CH$, it is evident that AM and

AN together equal the three sides, or the perimeter of the triangle ABC; and, since they are equal to each other, that AM alone equals the semiperimeter. And again, if FC be drawn, $\angle FCD$ is a right angle (s. Props. 13, 4), and so also is $\angle FBD$, and therefore the quadrilateral figure FCDB may be inscribed in a circle (iii. Prop. 22), of which FD is the diameter, and O, the middle point of FD, the centre. If then from O lines be drawn parallel to BC, and meeting in P and R, FL produced and DH, both perpendicular to BC, the triangles OFR and ODP will be equal (I. Prop. 26), and $OP = OR$; and if OK be drawn perpendicular to BC, then $KH = KL$ (I. Prop. 34). But $KB = KC$ (iii. Prop. 3), and therefore $KB - KH = KC - KL$ (Ax. 3), or $BH = CL$; and $BM = BH$, therefore $BM = CL = CE$ (s. Prop. 13). Now, since $BM = CL$, and $BG = BL$, it is evident that AG is the difference between AM or the semiperimeter and the side BC. And since $BM = CE$, and $AG = AE$, BG is the difference between the semiperimeter and the side AC, and it is obvious that BM is the difference between the semiperimeter and the side AB. Now, $AG \cdot AM : FG \cdot AM :: AG : FG$ (vi. Prop. 1); and also $FG \cdot AM : FG \cdot DM :: AM : DM$. But since the triangles AGF, AMD are similar, $AM : DM :: AG : FG$; therefore the three rectangles are in continued proportion, or $AG \cdot AM : FG \cdot AM :: FG \cdot AM : FG \cdot DM$. But the triangles FGB, BMD are also similar, since they are right-angled, and $\angle s$ GBF, MBD together equal a right angle; therefore $FG : GB :: BM : DM$, and $FG \cdot DM = GB \cdot BM$, and consequently $AG \cdot AM : FG \cdot AM :: FG \cdot AM : GB \cdot BM$; and since the perpendiculars drawn from F to the three sides of the triangle are equal (I. Prop. 26), the rectangle under FG and the semiperimeter is equal to the area of the triangle, and therefore, &c.

PROP. CXVIII.

Similar polygons inscribed in circles, or circumscribed about them, are to one another in the duplicate ratio of the rays of those circles.

It is evident, that in the inscribed polygons, the homologous sides subtend equal angles at the circumference (vi. Prop. 20), and therefore at the centre also (iii. Prop. 20); and consequently, rays drawn to the extremities of all the sides resolve the polygon into similar triangles, each homologous pair of which are to each other in the duplicate ratio of the rays of the circles (vi. Prop. 19). Therefore, all the triangles in each polygon, that is to say, the polygons themselves are to each other in the duplicate ratio of the rays of the circumscribing circles. The same thing may be proved of circumscribed polygons, by drawing rays to the points of contact.

PROP. CXIX.

The areas of any two circles are to each other in the duplicate ratio of their rays.

If from the extremities of any side of an inscribed polygon, lines be drawn to any point in the arch which it subtends, a triangle will thus be added to the polygon and taken from the circle. Consequently, the difference between a circle and the polygon is continually diminished as the number of the sides of the polygon is increased; and as they may be multiplied without end, it is manifest that a polygon may be inscribed in a circle, differing from it by less than any assignable difference. Then if it be supposed that two circles, A and a, are not in the duplicate ratio of their rays, R and r, but that $R^2 : r^2 :: A : a - d$; let similar polygons, P, p, be inscribed in those circles, such

that p may differ from a by a quantity less than d , or be greater than $a - d$. Then A will have a less ratio to p than it has to $a - d$, that is to say, than R^2 to r^2 ; therefore P , which is less than A , must have to p a less ratio than $R^2 : r^2$, but $P : p :: R^2 : r^2$ (s. Prop. 118); which is absurd; and therefore the circles must be in the duplicate ratio of their rays.

PROP. CXX.

The area of a circle is equal to a rectangle contained by its radius and a straight line equal to half its circumference.

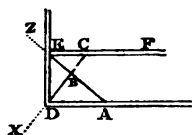
Let S be the semicircumference, and R the radius; and if it be supposed that the area of the circle is not equal to $S \cdot R$, but that it exceeds it by d , then it is evident that a polygon may be inscribed in the circle, differing from it by a quantity less than d , and therefore that the inscribed polygon is greater than $S \cdot R$: but the polygon is equal to the sum of the rectangles under the halves of its sides, (the sum of which halves is less than S), and the perpendiculars from the centre of the circle to those sides, which perpendiculars are less than R ; the polygon therefore is also less than $S \cdot R$; which is absurd. The area of the circle, therefore, does not exceed $S \cdot R$. Nor can it be less than $S \cdot R$, for then a circumscribed polygon might be in like manner proved to be both less and greater than $S \cdot R$. The area of the circle, therefore, is equal to the rectangle under its radius and semicircumference.

NOTE.

It is stated in the Appendix (E) to the Elements, that to find Two Mean Proportionals is a problem not admitting of a strictly geometrical solution;* but that Plato, among others, contrived to solve it by a tentative method; that is to say, by a method involving a construction which depends on experimental adaptation, and not attainable directly by means of the Postulates. The following solutions of this celebrated problem, by Plato, Philo of Byzantium, and Des Cartes, are worthy of notice.

PLATO'S METHOD.

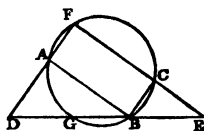
Let ADE be a square, or two rules fitted together at a right angle, and let EF be a rule standing on one of these, as DE, and moveable along it at right angles. Then place the given lines AB, CB together at a right angle, and apply them to the rules in such a manner, that while their extremities, A and C, are in contact with the rules, their prolongations, BZ, BX, may pass through the angles D and E. When this adjustment is effected, BD and BE are the required mean proportionals. For since ADE and DEC are both right-angled triangles, and BD, BE are their altitudes, it is evident, that $AB : BD :: BD : BE$ (vi. Prop. 8), and also that $BD : BE :: BE : BC$.



* In the place referred to (p. 180 of the 'Elements'), there is a slight typographical error, which might mislead the incautious. The four proportionals spoken of are in continued proportion, and therefore the expression $A : x :: y : B$ ought to be $A : x : y : B$, or more at length, $A : x :: x : y :: y : B$.

PHILO'S METHOD.

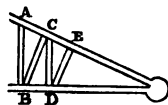
Place the given lines AB , BC at a right angle; complete the rectangle $ABCF$, and circumscribe a circle about it. Produce indefinitely FA , FC , and to the point B apply a rule, DE , in such a manner that BE , GD , the portions of it intercepted between the circumference of the circle and the produced lines, may be equal. This arrangement being made, the lines AD and CE are the required mean proportionals.



For since $BE = GD$, $GE = BD$, and consequently $GE \cdot EB = BD \cdot DG$. Therefore the rectangles $EF \cdot EC$, $DF \cdot AD$, which are respectively equal to those preceding (iii. Prop. 36), are equal to each other, and $EF : DF :: AD : EC$ (vi. Prop. 16). But owing to the similarity of the triangles EFD , BAD , ECB (vi. Prop. 2), $EF : DF :: AB : AD :: EC : CB$, and therefore $AB : AD :: AD : EC :: EC : CB$.

METHOD OF DES CARTES.

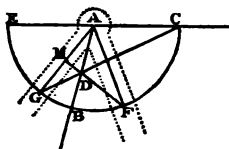
Insert in the arms of a rule having a hinged joint, alternate perpendiculars, AB , BC , CD , DE , each succeeding perpendicular passing through the right angle made by the preceding, and arrange them, by opening or closing the rule, so that the first, AB , and last, ED , may be equal to the given extremes. Then it is obvious, that on account of the similarity of the right-angled triangles, BC and CD will be the required mean proportionals, or that $AB : BC :: BC : CD :: CD : DE$.



By this specious method, it would appear that any number of mean proportionals may be found; but the necessary adjustments are almost as deficient in practical facility as in geometrical legitimacy.

Another problem, beyond the domain of Plane Geometry, viz. the TRISECTION OF AN ANGLE, admits of being solved by a tentative method, practically easy, and, we believe, not published nor generally known.

Let BAC be the given angle. Produce one of its legs, as AC , beyond the vertex, and with A as centre, describe the semicircle EBC . Place the vertex of an equilateral triangle of card or board on the point A , and let it move round the point on a pin. At M , the middle point of the leg AG of the triangle, or of the part intercepted between A and the circumference, fix a fine thread, and attach it to F ; and in like manner, fix a thread at G , and keep it extended through the point C . Now, if the triangle be moved round A , till the intersection D of the two threads falls on the line AB , the given angle will be trisected by AF , as is evident from S. Prop. 76.



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